

ABUNDANCE OF WILD HISTORIC BEHAVIOR, ERGODIC DECOMPOSITION AND GENERALIZED PHYSICAL MEASURES

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ABSTRACT. Using Caratheodory measures, we associate to each positive orbit $\mathcal{O}_f^+(x)$ of a measurable map f , a Borel measure η_x . We show that η_x is f -invariant whenever f is continuous or η_x is a probability. We use these measures to prove the Ergodic Decomposition Theorem and also to characterize the generic points of an invariant measure. These measures are used to study the *historic* points of the system, that is, *points with no Birkhoff averages*, and we construct topologically generic subset of *wild historic points* for wide classes of dynamical models. Using these tools, we extend the notion of physical measure and show that the generic homeomorphism has such a generalized physical measure. We use properties of the measure η_x to deduce some features of the dynamical system involved, like the *existence of heteroclinic connections from the existence of open sets of historic points*.

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1. INTRODUCTION

The asymptotic behavior of a dynamical system given by some transformation $f : M \rightarrow M$ is in general quite complex. To understand the behavior of the orbit of a point x we should focus on the ω -limit set $\omega(x)$, the set of accumulation points of $f^n(x)$, $n \geq 1$. The number of iterates of the positive orbit of x near some given point $y \in \omega(x)$ depends on y and, in fact, the limit frequency of visits of the orbit of x to some subset A of the phase space M is non-existent in general!

A simple and very general example is obtained taking a non-periodic point x , an strictly increasing integer sequence $n_j = \sum_{i=0}^j 10^i$ and the subset

$$A = \{f^{n_1}(x), \dots, f^{n_2}(x)\} \cup \{f^{n_3}(x), \dots, f^{n_4}(x)\} \cup \dots = \bigcup_{i \geq 0} \{f^{n_{2i+1}}(x), \dots, f^{n_{2(i+1)}}(x)\}.$$

Then it is straightforward to check that

$$\frac{1}{n_k} \sum_{j=1}^{n_k} \delta_{f^j(x)}(A) \begin{cases} \geq 9/10 & \text{if } k \text{ is even} \\ \leq 1/10 & \text{if } k \text{ is odd} \end{cases}.$$

Thus even if $\lim_k f^k(x)$ exists, we still have that $\lim_k \frac{1}{n_k} \sum_{j=1}^{n_k} \delta_{f^j(x)}(A)$ does not exist. In Ergodic Theory behavior of this type is bypassed through the use of weak* convergence in Birkhoff's Ergodic Theorem, ensuring that $\lim_k \frac{1}{n_k} \sum_{j=1}^{n_k} \delta_{f^j(x)}$ exists in the weak* topology for almost every point with respect to any f -invariant measure.

As defined by Ruelle in [50] and Takens in [56] we say that x has “*historic behavior*” if the sequence $\frac{1}{n_k} \sum_{j=1}^{n_k} \delta_{f^j(x)}$ does not converge in the weak* topology.

The above construction of non-convergent sequence of frequency of visits can be easily obtained in any topological Markov chain and so, through Markov partitions and their coding, can also be obtained in every hyperbolic basic set of an Axiom A diffeomorphism; see e.g. Bowen and Ruelle in [17].

As shown by Takens [56] (and also Dowker [21]) the set of points with historic behavior is residual. Hence, generically the points of a basic piece of an Axiom A diffeomorphism has *both dense orbit and historic behavior!* More recently, Liang-Sun-Tian [35] extend this

result for the support of non-uniformly hyperbolic measures for smooth diffeomorphisms and Kiriki-Ki-Soma [32] obtain a residual historic subset in the basin of attractor of any geometric Lorenz model. Here we extend and strengthen these results to whole classes of dynamical models.

We say that a measure μ which gives positive mass to the subset of points with historic behavior is a *historic measure*.

The construction of the heteroclinic attractor attributed to Bowen, as presented in [55], as a flow with an open subset of points with historic behavior, provides an example of such phenomenon where Lebesgue measure is a non-invariant historic measure; see Example 2 in Section 3. Other similar examples can be found in Gaunersdorfer [22]. In the quadratic family the following result from Hofbauer-Keller provides many other examples.

Theorem 1.1 ([28]). *For the family $f_t : [0, 1] \circlearrowleft, f_t(x) = 4tx(1 - x)$, there exists a non-denumerable subset H of parameters in $[0, 1]$ such that Lebesgue measure is an historic measure with respect to f_t for all $t \in H$.*

We emphasize that Jordan, Naudot and Young showed in [29], through a classical result from Hardy [26], that *if time averages $\frac{1}{n} \sum_{j=0}^{n-1} \phi(f^j x)$ of a bounded observable $\phi : \mathbb{X} \rightarrow \mathbb{R}$ do not converge, then all higher order averages do not exist either, that is, every Césaro or Hölder higher order means fail to converge*; see e.g. [27] for the definitions of these higher order summation processes. This shows that *historic points cannot be regularized by taking higher order averages*.

This paper aims at studying the ergodic properties of historic measures. For that we consider a set function $\tau_x(A) = \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \delta_{f^j(x)}(A)$ that will serve as a premeasure to obtain a Borel measure η_x through a classic well-known construction of Carathéodory; a thorough presentation of which can be found in Rogers [49]. This is a way to bypass failure of convergence of time averages in certain classes of systems, in particular in the “Bowen eye” example.

The following Block and Lyubich result provides a natural notion of ergodicity for non-invariant measures.

Theorem 1.2 ([14]). *Lebesgue measure is ergodic (but non-invariant) for every S -unimodal transformation (in particular for the quadratic family) which has no periodic attractor.*

Using this notion we obtain: *for an ergodic measure μ (not necessarily invariant) $\eta_x = \eta_y$ for μ -almost every x, y* . In addition, for the quadratic family f_t there exists a f_t -invariant measure μ_t such that $\eta_x = \mu_t$ for Lebesgue almost every x .

We also reprove the Ergodic Decomposition Theorem (see e.g. Mañé [36]) using η_x as ergodic components with respect to every f -invariant probability measure.

These measures η_x can be used to define an extension of the notion of physical measure, and using a recent result from Abdenur and Andersson [1] we show that these *generalized physical measures* do exist for the generic homeomorphism of any compact boundaryless manifold.

We show that if η_x is purely atomic for an open subset of points x , then the system has similar dynamics to the heteroclinic Bowen attractor; see [55, 22]. In particular Lebesgue measure is historic.

Many more results about the set of points with historic behavior are known. To the best of our knowledge, Pesin and Pitskel were the first to show that these points carry full topological pressure and satisfy a variational principle for full shifts, in [46]. These points are named “non-typical” in [12] by Barreira-Schmeling who show that this set has full Hausdorff dimension and full topological entropy for subshifts of finite type, conformal repellers and conformal horseshoes. These points are said “irregular points” in the work [57] of Thompson where it is shown that, for maps with the specification property having some point with historic behavior, the set of such irregular points carries full topological pressure. This indicates that this set of points is not dynamically irrelevant. Genericity of these “irregular points” for dynamics with specification and non-convergence of time averages for an open and dense family of continuous functions follows from [20, Proposition 21.18]; see Lemma 5.4 in Subsection 5.1.3 and also [9, 7, 8, 6].

The Hausdorff dimension of subsets of such points is studied by Barreira-Saussol [11] for hyperbolic sets for smooth transformations and also by Olsen-Winter in [43] in the setting of multifractal analysis for subshifts of finite type and for several specific transformations also by Olsen [41, 40] and generalized by Zhou-Chen [59].

Recently Kiriki and Soma [33] show that every C^2 surface diffeomorphism exhibiting a generic homoclinic tangency is accumulated by diffeomorphisms which have non-trivial wandering domains whose forward orbits have historic behavior. More recently Laboriau and Rodrigues [34] motivated by the ideas in [33] present an example of a parametrized family of flows admitting a dense subsets of parameter values for which the set of initial conditions with historic behaviour contains an open set. These are examples of persistent historic behavior.

In this paper we also provide broad classes of examples where η_x has infinite mass on every open subset. These *points with wild historic behavior* are also topologically generic for Axiom A systems, for expanding measures, topological Markov chains, suspension semi-flows over these maps, Lorenz-like attractors and many other dynamical models. These are the “points with maximal oscillation” studied by Denker, Grillenberger and Sigmund in [20, Proposition 21.18]; see Lemma 5.7 in Subsection 5.1.4. Olsen in [42, 39] studied “extreme non-normal” numbers and continued fractions which are, in particular, wild historic points, and so form a topologically generic subset; see Example 3 in Section 5.1.

We note that there are classes of systems with no specification where our construction can be performed: see Example 5 and Remark 5.12 in Subsection 5.7. Hence, the genericity of wild historic points is more general than the genericity of points with maximal oscillation in systems with specification.

2. STATEMENT OF RESULTS

2.1. Measure associated to a sequence of outer measures. Let \mathbb{X} be a compact metric space, $f : \mathbb{X} \rightarrow \mathbb{X}$ a measurable map with respect to the Borel σ -algebra \mathfrak{B} and let

dist be the distance on \mathbb{X} . Let $(\xi_j)_{j \in \mathbb{N}}$ be a sequence of finitely additive outer probabilities on \mathbb{X} , so that each ξ_j is a set function defined on all parts of \mathbb{X} with values in $[0, 1]$ and such that $\xi_j(\mathbb{X}) = 1 \ \forall j$.

Definition 1. Given a sequence $(\xi_j)_{j \in \mathbb{N}}$ of finitely additive outer measures with unit total mass, we define

$$\tau(A) = \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \xi_j(A).$$

It is easy to see that

$$\tau(A) \leq \tau(A \cup B) \leq \tau(A) + \tau(B)$$

for all $A, B \subset \mathbb{X}$ and consequently

$$\tau\left(\bigcup_{j=1}^n A_j\right) \leq \sum_{j=1}^n \tau(A_j), \quad (1)$$

for all finite collection A_1, \dots, A_n of subsets of \mathbb{X} .

However, one can easily find examples of countable collections $(A_j)_{j \in \mathbb{N}}$ of sets and of measures $(\xi_j)_{j \in \mathbb{N}}$ such that

$$\tau\left(\bigcup_{j \in \mathbb{N}} A_j\right) > \sum_{j \in \mathbb{N}} \tau(A_j).$$

Example 1. If the positive f -orbit $\mathcal{O}^+(x) = \{x_j = f^j(x) : j \in \mathbb{N}\}$ is infinite then, taking $A_j = \{x_j\}$ and ξ_j the Dirac point mass at x_j , i.e. $\xi_j = \delta_{x_j}$, we get $\tau(A_j) = 0$ for each $j \geq 1$ and so $1 = \tau(\mathcal{O}^+(x)) = \tau(\cup A_j) > \sum_j \tau(A_j) = 0$.

Let \mathcal{A} be the collection of open sets of \mathbb{X} . According to the definition of pre-measure of Rogers [49], τ restricted to \mathcal{A} is a *pre-measure*; see [49, Definition 5].

Definition 2. Given $Y \subset \mathbb{X}$, define

$$\nu(Y) = \sup_{r > 0} \nu_r(Y) \quad \left(= \lim_{r \searrow 0} \nu_r(Y) \right),$$

where $\nu_r(Y) = \inf_{\mathcal{I} \in \mathcal{A}(r, Y)} \sum_{I \in \mathcal{I}} \tau(I)$ and $\mathcal{A}(r, Y)$ is the set of all countable covers $\mathcal{I} = \{I_i\}$ of Y by elements of \mathcal{A} with $\text{diam}(I_i) \leq r \ \forall i$.

The function ν , defined on the class of all subset of \mathbb{X} , is called in [49] the (*Caratheodory*) *metric measure constructed from the pre-measured τ by Method II* (Theorem 15 of [49]).

We define η to be the restriction of ν to the Borel sets, i.e., $\eta = \nu \mid \mathfrak{B}$. From [49, Theorem 23] (see also [49, Theorem 3]) it follows that η is a countable additive measure defined on the σ -algebra \mathfrak{B} of Borel sets.

Certain measure-theoretical properties of η and those of τ are provided in Section 3: first with no dynamical assumptions, and then assuming that τ is f -invariant.

2.2. Ergodic decomposition with measures generated by an orbit. Considering the particular choice $\xi_j = \delta_{x_j}$ with $x_j = f^j(x_0)$, $j \geq 0$ for a given $x_0 \in \mathbb{X}$, it is easy to see that the associated set function τ satisfies $\tau(f^{-1}(A)) = \tau(A)$ for every subset A of \mathbb{X} . We denote by $\eta_x = \eta$ the measure obtained from τ with the above choices.

Using the properties of τ versus η we prove the following version of the Ergodic Decomposition Theorem in Section 4.2.

Theorem A. *Let \mathbb{X} be a perfect compact metric space and let $f : \mathbb{X} \rightarrow \mathbb{X}$ be a measurable map. If μ is an f -invariant finite measure then, for μ almost every $x \in \mathbb{X}$, η_x is a f -invariant ergodic probability. Furthermore, $\mu = \int_{x \in \mathbb{X}} \eta_x d\mu$. That is,*

$$\int \varphi d\mu = \int_{x \in \mathbb{X}} \left(\int \varphi d\eta_x \right) d\mu$$

for every $\varphi \in L^1(\mu)$ (in particular, $\mu(V) = \int_{x \in \mathbb{X}} \eta_x(V) d\mu$ for every Borel set $V \subset \mathbb{X}$).

This is similar to [36, Theorem 6.2] but our proof is different and our approach provides a natural generalization of the notion of physical measure.

2.3. Wild historic behavior generically.

Example 2. We consider the well-known example of a planar flow with divergent time averages attributed to Bowen; see Figure 1 and [55]. We set $f = X_1$ the time-1 map of the flow, and define $\xi_j = \delta_{x_j}$ with $x_j = f^j(x_0)$ for $j \geq 1$, where x_0 is a point in the interior of the plane curve formed by the heteroclinic orbits connecting the fixed hyperbolic saddle points A, B .

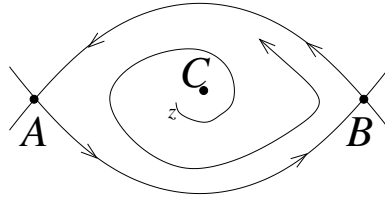


FIGURE 1. A planar flow with divergent time averages

We assume that this cycle is attracting, that is, the homoclinic connections are also the set of accumulation points of the positive orbit of x_0 ; in particular, we have $|\det Df| < 1$ near the stable/unstable manifold of the saddles A, B . However, this accumulation is highly unbalanced statistically.

Indeed, if we denote the expanding and contracting eigenvalues of the linearized vector field at A by α_+ and α_- and at B by β_+ and β_- , and the modulus associated to the upper and lower saddle connection by

$$\lambda = \alpha_-/\beta_+ \quad \text{and} \quad \sigma = \beta_-/\alpha_+$$

then $\lambda > 0$, $\sigma > 0$ and $\lambda\sigma > 1$, since the cycle is assumed to be attracting; see [55], where it is proved that for every continuous function $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$ with $\varphi(A) > \varphi(B)$ we have

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(x_j) &= \frac{\sigma}{1+\sigma} \varphi(A) + \frac{1}{1+\sigma} \varphi(B), \quad \text{and} \\ \liminf_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(x_j) &= \frac{\lambda}{1+\lambda} \varphi(B) + \frac{1}{1+\lambda} \varphi(A). \end{aligned}$$

In particular, for any small $\varepsilon > 0$

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \chi_{B_\varepsilon(A)}(x_j) \geq \frac{\sigma}{1+\sigma} \quad \text{and} \quad \limsup_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \chi_{B_\varepsilon(B)}(x_j) \geq \frac{\lambda}{1+\lambda},$$

so that $\tau(B_\varepsilon(A)) \geq \sigma(1+\sigma)^{-1}$ and $\tau(B_\varepsilon(B)) \geq \lambda(1+\lambda)^{-1}$ for all $\varepsilon > 0$. Since we can find $\varepsilon > 0$ arbitrarily small such that $B_\varepsilon(A), B_\varepsilon(B)$ are in \mathcal{G} , then we deduce

$$2 > \eta_{x_0}(\mathbb{X}) = \eta_{x_0}(A) + \eta_{x_0}(B) = \frac{\sigma}{1+\sigma} + \frac{\lambda}{1+\lambda} = \frac{\sigma + \lambda + 2\lambda\sigma}{1+\sigma + \lambda + \lambda\sigma} > 1$$

because $\lambda\sigma > 1$.

Moreover we also obtain $\limsup \frac{1}{n} \log \|Df^n(x)\| > 0 > \liminf \frac{1}{n} \log \|(Df^n(x))^{-1}\|$ for Lebesgue almost every point x , so that there is no Physical/SRB measure.

Definition 3. We say that $x \in \mathbb{X}$ for which η_x is not a probability measure is a *historic point* or a point with *historic* behavior. We denote the set of historic points of the map f by $\mathcal{H} = \mathcal{H}_f$.

This definition follows the one in Takens [56]. The existence of such points can be obtained in any compact invariant set of a map which is conjugate to a full shift, e.g. an horseshoe. It can be easily adapted to any topological Markov chain (Markov subshift of finite type) and hence can be applied to any basic set of an Axiom A diffeomorphism; see e.g. [53, 17, 16] for the definitions of Axiom A diffeomorphisms and [56].

In fact, Dowker [21] showed that if a transitive homeomorphism of a compact set \mathbb{X} admits a point z with transitive orbit and historic behavior, then historic points form a topologically generic subset of \mathbb{X} . As observed in [56], the generic subset of points with historic behavior exists also in the stable set of any basic set Λ of an Axiom A diffeomorphism.

Combining the construction in [56] with density of hyperbolic points we can obtain a stronger property for a topologically generic subset points.

Definition 4. We say that $x \in \mathbb{X}$ for which η_x gives infinite mass to every open subset of a compact invariant subset Λ for the dynamics, is a point with *wild historic behavior* in Λ or a *wild historic point*. We denote the set of wild historic points of the map f by $\mathcal{W} = \mathcal{W}_f$.

The following result provides plenty of classes of examples of abundance of wild historic behavior.

Theorem B. *The set of points with wild historic behavior in*

- (1) *every mixing topological Markov chain with a denumerable set of symbols (either one-sided or two-sided);*
- (2) *every open continuous transitive and positively expansive map of a compact metric space;*
- (3) *each local homeomorphism defined on an open dense subset of a compact space admitting an induced full branch Markov map;*
- (4) *suspension semiflows, with bounded roof functions, over the local homeomorphisms of the previous item;*
- (5) *the stable set of any basic set Λ of either an Axiom A diffeomorphism, or a Axiom A vector field;*
- (6) *the support of an expanding measure for a C^{1+} local diffeomorphism away from a non-flat critical/singular set on a compact manifold;*
- (7) *the support of a non-atomic hyperbolic measure for a C^{1+} diffeomorphism, or a C^{1+} vector field, of a compact manifold;*

is a topologically generic subset (denumerable intersection of open and dense subsets).

We stress that item (2) provides mild conditions to obtain a generic subset of wild historic points: positively expansive maps f are those for which there exists $\delta > 0$ so that any pair x, y of distinct points are guaranteed to be δ -apart in some finite time, that is, there exists $N = N(x, y) \geq 0$ so that $d(f^N x, f^N y) > \delta$ (also known as “sensitive dependence”). This property together with the existence of a dense orbit in a compact metric space ensures the existence of plenty wild historic points.

For detailed definitions of positively expansive, induced full branch Markov map, expanding measure, Axiom A systems and hyperbolic measures, see Section 5 and references therein.

The connection between properties of real numbers in the interval $[0, 1]$ and properties of the orbits of the maps $T_b : [0, 1] \ni x \mapsto bx \bmod 1$ for any integer $b \geq 2$ and the Gauss map $G : [0, 1] \ni x \mapsto \frac{1}{x} \bmod 1$, together with the natural coding of the dynamics by topological Markov chains, enables us to easily relate wild historic behavior with absolutely abnormal numbers and extremely non-normal continued fractions; see e.g. [40, 41, 43] and references therein. Theorem B contains some of the genericity results in these works as particular cases; see e.g. Example 3 in Section 5.1.

We have obtained a topologically generic subset of wild historic points for abundant classes of systems with (non uniform) hyperbolic behavior. This indicates that *the absence of wild historic points implies that all invariant probability measures are either atomic or non-hyperbolic*, that is, no wild historic behavior forces every non-atomic invariant probability measure to have zero Lyapunov exponents, either for diffeomorphisms, vector fields or endomorphisms of compact manifolds; and even for local diffeomorphisms away from a singular set. The only extra assumptions being sufficient smoothness (Hölder- C^1 seems to be enough) and a singular/critical set regular enough. We present some conjectures in Subsection 2.6.

2.4. Generalized physical measures. Let $f : \mathbb{X} \rightarrow \mathbb{X}$ be a measurable map defined on a compact metrical space \mathbb{X} . Given a reference measure μ , we may ask if there is an invariant measure ν such that

$$\lim_n \frac{1}{n} \sum_{j=0}^{+\infty} \varphi \circ f^j(x) = \int \varphi d\nu \quad (2)$$

for every continuous function φ and every point in some set $U \subset \mathbb{X}$ with $\mu(U) > 0$.

More generally, we may ask if there is a set $U \subset \mathbb{X}$ with $\mu(U) > 0$ and an invariant measure ν so that

$$\int_U \left(\lim_n \frac{1}{n} \sum_{j=0}^{+\infty} \varphi \circ f^j(x) \right) d\mu = \int \varphi d\nu \quad (3)$$

for every continuous function φ .

If \mathbb{X} is a Riemann compact manifold and the reference measure is the Lebesgue measure, i.e. $\mu = \text{Leb}$, a measure ν satisfying (2) above is called a *physical measure*. Moreover, if $\nu \ll \text{Leb} = \mu$ (or ν is absolutely continuous with respect to the Lebesgue measure on the unstable leaves), ν is called a *SRB measure*.

Definition 5. We say that ν is a μ -*physical measure* if (2) holds for a set with μ positive measure.

Definition 6. We say that ν is a *generalized μ -physical measure* if (3) holds for a set U with $\mu(U) > 0$.

As in the invariant case, a non invariant measure μ is called *ergodic* if $\mu(U) = 0$ or $\mu(\mathbb{X} \setminus U) = 0$ for every invariant set U (i.e., $f^{-1}(U) = U$). There are many examples of non invariant ergodic measure. For instance, Martens [37] proved that the Lebesgue measure is ergodic for every S -unimodal map without periodic attractor. More generally, it was shown in [47] that every (non invariant) expanding measure with bounded distortion can be decomposed into a finite number of (non invariant) ergodic measures.

The following provides a necessary and sufficient condition for existence of generalized μ -physical measures.

Theorem C. *Let $f : \mathbb{X} \rightarrow \mathbb{X}$ be continuous. Then f admits a generalized μ -physical measure if, and only if, $\mu(\{x \in \mathbb{X}; \eta_x(\mathbb{X}) = 1\}) > 0$. If, in addition, μ is ergodic (but not necessarily invariant), then f admits a μ -physical measure.*

We prove this in Section 6.1 after studying the behavior of the measure η in the setting of continuous dynamics, done in Section 6. In Section 6.2 we obtain necessary and sufficient conditions for the existence of a μ -physical measure.

The notion of generalized physical measure has some resemblances with the notions of “Iliashenko’s statistical attractors”, “Milnor’s attractors” and “SRB-like measures”; see [19, 18] and references therein. However, our definition is not restricted to using Lebesgue measure as a reference measure and our construction is significantly different.

2.4.1. *Generalized physical measures exist for the generic homeomorphism.* Recently Abdenur and Andersson [1] show that

Theorem 2.1. *For the generic homeomorphism $f : M \rightarrow M$ of a compact boundaryless finite manifold Birkhoff time averages*

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x))$$

exist Lebesgue almost everywhere for every continuous function $\varphi : M \rightarrow \mathbb{R}$.

From the properties of τ and η obtained for continuous dynamics in Section 6, we see that Theorem 2.1 ensures that, for a generic subset of homeomorphisms $f : M \rightarrow M$, the measure η_x is a f -invariant probability measure for Lebesgue almost every x . Hence we deduce

Corollary D. *The generic homeomorphism $f : M \rightarrow M$ of a compact boundaryless finite manifold admits generalized Lebesgue-physical measures.*

This shows that our definition of generalized physical measure is relevant for a large family of continuous dynamics.

In Example 6 of Section 6 we provide an example of a smooth systems with no Lebesgue-generalized physical measure.

2.5. Strong historic behavior and heteroclinic attractors. The following result shows that strong historic behavior in the neighborhood of two hyperbolic periodic points is sufficient to obtain an heteroclinic connection relating the two points.

Theorem E. *Let $f : M \rightarrow M$ be a C^1 diffeomorphism on a compact boundaryless manifold M endowed with a pair of hyperbolic periodic points P, Q satisfying, for some $\varepsilon > 0$*

- (H) *η_x is atomic with two atoms P, Q for every x either in $B_\varepsilon(P) \setminus (W_\varepsilon^s(P) \cup W_\varepsilon^u(P))$ or $B_\varepsilon(Q) \setminus (W_\varepsilon^s(Q) \cup W_\varepsilon^u(Q))$.*

Then P and Q have a heteroclinic cycle: $W^u(P) = W^s(Q)$ and $W^s(P) = W^u(Q)$.

That is, this result gives a sufficient condition for a diffeomorphism to exhibit an heteroclinic attractor, as in the example of Bowen as presented in [55] and Example 2.

2.6. Conjectures. We believe that it is possible to use properties of the measures η_x to understand certain dynamical features of the system involved.

As a simple example, we observe that $\text{Per}(f) = \emptyset$ for a continuous map $f : \mathbb{X} \rightarrow \mathbb{X}$ of a compact space, implies $\eta_x(\mathbb{X}) < \infty$ for all $x \in \mathbb{X}$ (e.g., an irrational circle rotation or torus translation).

Indeed, given $m > 1$, every point $y \in \mathbb{X}$ admits a neighborhood $U_{y,m}$ such that $U_{y,m} \cap f^i(U_{y,m}) = \emptyset, i = 1, \dots, m$ and so $\eta_x(U_{y,m}) < 1/m$ for every $x \in \mathbb{X}$. We obtain an open cover $\{U_{y,m}\}_{y \in \mathbb{X}}$ of the compact \mathbb{X} and so a finite subcover U_1, \dots, U_k exists. Hence $\eta_x(\mathbb{X}) \leq \sum_{i=1}^k \eta_x(U_i) \leq k/m < \infty$.

Another observation is that if $\eta_x(\mathbb{X}) < \infty$ for all $x \in \mathbb{X}$, then $h_{top}(f) = 0$ for a diffeomorphism $f : S \rightarrow S$ of a compact surface or a endomorphism $f : I \rightarrow I$ of the circle or interval.

Indeed, in those cases, $h_{top}(f) > 0$ ensures the existence of a horseshoe Λ for f ; see [30]. In both cases, these invariant subsets are conjugated to a full shift with finitely many symbols, and so if η_x is always a finite measure we contradict Theorem B.

It is then natural to pose the following

Conjecture 1. Every smooth (C^{1+}) diffeomorphism $f : \mathbb{X} \rightarrow \mathbb{X}$ of a compact manifold \mathbb{X} satisfying $\eta_x(\mathbb{X}) < \infty$ for all $x \in \mathbb{X}$ has zero topological entropy.

Note that the examples provided by Beguin, Crovisier and Le Roux in [13] show that this conjecture is false if we allow the dynamics to be just a homeomorphism.

It is known in many cases that points with historic behavior form a geometrically big subset (full Hausdorff dimension) of the dynamics; see e.g. Barreira-Schmeling [12]. Recently Zhou and Chen [60] show that the set of historic points carries full topological pressure for systems with non-uniform specification under certain conditions. Here we have shown that wild historic points are generic in a broad class of examples.

Conjecture 2. In the class of examples considered in Theorem B the set of wild historic points has full Hausdorff dimension, full topological entropy and full topological pressure.

Developing our observation about absence of wild historic behavior after Theorem B, we propose the following.

Conjecture 3. Absence of wild historic points for a smooth enough (C^{1+}) dynamics of a diffeomorphism or a vector field in a compact manifold implies that all invariant probability measures are either atomic or have only zero Lyapunov exponents.

An analogous conclusion holds for all smooth enough (C^{1+}) local diffeomorphisms away from a critical/singular set which is sufficiently regular (non-flat).

Note that from Peixoto's Theorem [45, 44, 24, 25] for an open and dense subset of vector fields in the C^r topology on compact orientable surfaces, for all $r \geq 1$, the limit set of every orbit is contained in one of finitely many hyperbolic critical elements (fixed points or periodic orbits). Hence wild historic points are absent from an open and dense subset of smooth continuous time dynamics on surfaces. Thus, for vector fields Conjecture 3 makes sense only on manifolds of dimension 3 or higher.

We propose the following weakening of Condition (H) from Theorem E.

Conjecture 4. For a C^r diffeomorphism of a compact manifold of dimension 2 or higher, $r \geq 1$, if there exists a point x and two hyperbolic periodic points P, Q and $a, b \geq 0, a+b > 1$ so that

$$\eta_x = a \cdot \frac{1}{\pi(P)} \sum_{j=1}^{\pi(P)} \delta_{f^j P} + b \cdot \frac{1}{\pi(Q)} \sum_{j=1}^{\pi(Q)} \delta_{f^j Q},$$

where $\pi(P), \pi(Q)$ give the minimal periods of P, Q , then f is accumulated in the C^r topology by diffeomorphisms g so that the continuations P_g, Q_g of the periodic points P, Q are homoclinically related.

2.7. Organization of the text. We present the construction of the measure η through a pre-measure τ in a very general dynamical setting in Section 3. Then we specify the construction of the measure η_x for a given orbit of a point x in Section 4, where we also deduce Theorem A on ergodic decomposition of invariant measures using such frequency measures η_x . We construct the generic subset of wild historic points in Section 5 proving Theorem B and providing abundant classes of examples to apply our results. In Section 6, we consider continuous transformations and relate our constructions with the notion of physical measure and extend this to the notion of generalized physical measure, proving Theorem C and Corollary D. Finally, in Section 7 we prove Theorem E.

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3. THE MEASURE η VERSUS THE PREMEASURE τ

Here we provide some properties of the premeasure τ and of the measure η without assuming invariance of τ .

Lemma 3.1. *If \mathcal{K} is a uncountable collection of pairwise disjoint compact sets of \mathbb{X} , then there is a countable sub-collection $\mathcal{K}_0 \subset \mathcal{K}$ such that $\lim_{\varepsilon \rightarrow 0} \tau(B_\varepsilon(K)) = 0$ for every $K \in \mathcal{K} \setminus \mathcal{K}_0$.*

Proof. Let \mathcal{K}_n be the family of $K \in \mathcal{K}$ such that $\frac{1}{n} < \lim_{\varepsilon \rightarrow 0} \tau(B_\varepsilon(K)) \leq \frac{1}{n-1}$, for $n > 1$, and let $\mathcal{K}_0 = \bigcup_{n \geq 1} \mathcal{K}_n$. If \mathcal{K}_0 is uncountable, then there is some $n \geq 1$ such that \mathcal{K}_n is uncountable. As $\varepsilon \mapsto \tau(B_\varepsilon(K))$ is an increasing function, it follows that $\tau(B_\varepsilon(K)) > 1/n$ for all $K \in \mathcal{K}_n$ and all $\varepsilon > 0$. Set $\mathcal{K}^1 = \mathcal{K}_n$ and let \mathcal{K}_j^1 be the collection of $K \in \mathcal{K}^1$ such that $\frac{1}{j} \sum_{i=0}^{j-1} \xi_i(B_\varepsilon(K)) > \frac{1}{2n}$ for each $\varepsilon > 0$.

As \mathcal{K}^1 is uncountable, there are infinitely many $j \in \mathbb{N}$ such that \mathcal{K}_j^1 is uncountable. Thus, let s be such that $\mathcal{K}^2 = \mathcal{K}_s^1$ is uncountable and, for each $j \in \mathbb{N}$, let \mathcal{K}_j^2 be the collection of $K \in \mathcal{K}^2$ such that $\frac{1}{s} \sum_{i=0}^{s-1} \xi_i(B_\varepsilon(K)) > \frac{1}{2n}$ for each $\varepsilon > 0$.

Let $K_1, K_2, \dots, K_{4n} \in \mathcal{K}^2$ be any finite collection of elements of \mathcal{K}^2 with $K_i \neq K_t$ for $i \neq t$. Let also $\varepsilon > 0$ be such that $B_\varepsilon(K_i) \cap B_\varepsilon(K_t) = \emptyset$ for $i \neq t$. For $L = K_1 \cup \dots \cup K_{4n}$ we have

$$1 \geq \frac{1}{s} \sum_{i=0}^{s-1} \xi_s(L) = \frac{1}{s} \sum_{t=0}^{4n-1} \sum_{i=0}^{s-1} \xi_i(B_\varepsilon(K_t)) > 4n \frac{1}{2n} = 2$$

a contradiction. Hence \mathcal{K}_0 is countable and, as $\lim_{\varepsilon \rightarrow 0} \tau(B_\varepsilon(K)) = 0$ for every $K \in \mathcal{K} \setminus \mathcal{K}_0$ the proof is complete. \square

Lemma 3.2. *If $K \subset \mathbb{X}$ is a compact set, then $\eta(K) = 0 \iff \lim_{\varepsilon \rightarrow 0} \tau(B_\varepsilon(K)) = 0$.*

Proof. First suppose that $\eta(K) = 0$. Given $\delta > 0$ let \mathcal{C} be an open cover of K such that $\sum_{A \in \mathcal{C}} \tau(A) < \delta$. As K is compact there is a finite subcover $\mathcal{C}' \subset \mathcal{C}$ of K . Of course, $\sum_{A \in \mathcal{C}'} \tau(A) \leq \sum_{A \in \mathcal{C}} \tau(A) < \delta$. Let $\varepsilon_0 > 0$ be such that $B_{\varepsilon_0}(K) \subset \bigcup_{A \in \mathcal{C}'} A$. As \mathcal{C}' is finite, we get $\tau(B_\varepsilon(K)) \leq \tau(\bigcup_{A \in \mathcal{C}'} A) \leq \sum_{A \in \mathcal{C}'} \tau(A) < \delta \forall 0 < \varepsilon \leq \varepsilon_0$. As a consequence, $\lim_{\varepsilon \rightarrow 0} \tau(B_\varepsilon(K)) = 0$.

Now, suppose that $\lim_{\varepsilon \rightarrow 0} \tau(B_\varepsilon(K)) = 0$. Let $\mathcal{C} \in \mathcal{A}(r, K)$ be a finite cover of K . Given $\delta > 0$, let $\varepsilon > 0$ be such that $\tau(B_\varepsilon(K)) < \frac{\delta}{\#\mathcal{C}}$. Let $\mathcal{C}' = \{A \cap B_\varepsilon(K) ; A \in \mathcal{C}\}$. Note that $\mathcal{C}' \in \mathcal{A}(r, K)$ and $\sum_{A \in \mathcal{C}'} \tau(A) = \sum_{A \in \mathcal{C}} \tau(A \cap B_\varepsilon(K)) \leq \sum_{A \in \mathcal{C}} \tau(B_\varepsilon(K)) < \#\mathcal{C} \frac{\delta}{\#\mathcal{C}} = \delta$. Thus, $\nu_r(K) = 0 \forall r > 0$ and, as a consequence, $\eta(K) = 0$. \square

Corollary 3.3. *If \mathcal{K} is a uncountable collection of pairwise disjoint compact subsets of \mathbb{X} , then there is a countable sub-collection $\mathcal{K}_0 \subset \mathcal{K}$ such that $\eta(K) = 0$ for every $K \in \mathcal{K} \setminus \mathcal{K}_0$.*

Lemma 3.4. *If $A \subset \mathbb{X}$ is an open set with $\eta(\partial A) = 0$ then $\tau(\overline{A}) \leq \eta(A)$.*

Proof. Let $\delta > 0$ be a positive number. As η is regular, let $\varepsilon_0 > 0$ be small so that $\eta(K_{\varepsilon_0}) > \eta(A) - \delta/2$, where $K_\varepsilon = A \setminus B_\varepsilon(\partial A)$. By lemma 3.2, we can take $0 < \varepsilon < \varepsilon_0/2$ so that $\tau(B_{2\varepsilon}(\partial A)) < \delta/2$.

Let $r > 0$ be small enough so that $\nu_r(A) \leq \eta(A) \leq \nu_r(A) + \delta/2$. Let $\mathcal{C} \in \mathcal{A}(r, A)$ be such that $\nu_r(A) \leq \sum_{V \in \mathcal{C}} \tau(V) \leq \nu_r(A) + \delta/2$. Thus, $\eta(A) \leq \sum_{V \in \mathcal{C}} \tau(V) + \delta/2 \leq \eta(A) + \delta$, or equivalently,

$$\left| \eta(A) - \sum_{V \in \mathcal{C}} \tau(V) \right| \leq \delta/2.$$

As $\mathcal{C}_1 = \{V \cap A ; V \in \mathcal{C}\} \in \mathcal{A}(r, V)$ and $\sum_{V \in \mathcal{C}_1} \tau(V) \leq \sum_{V \in \mathcal{C}} \tau(V)$, changing \mathcal{C} by \mathcal{C}_1 if necessary, we may assume that $V \subset A \forall V \in \mathcal{A}(r, A)$.

Now since K_ε is compact, there is some finite open cover of $\mathcal{C}' \subset \mathcal{C}$ of K_ε . Therefore, $A' := \bigcup_{V \in \mathcal{C}'} V \subset A$ and $\sum_{V \in \mathcal{C}'} \tau(V) \leq \sum_{V \in \mathcal{C}} \tau(V) < \eta(A) + \delta/2$. As \mathcal{C}' is finite, we get $\tau(A') \leq \sum_{V \in \mathcal{C}'} \tau(V)$ and so,

$$\tau(A') \leq \eta(A) + \delta/2.$$

As $\overline{A} \subset A' \cup B_{2\varepsilon}(\partial A)$, we get $\tau(\overline{A}) \leq \tau(A') + \delta/2$. As a consequence,

$$\tau(\overline{A}) \leq \eta(A) + \delta$$

for every $\delta > 0$. \square

In what follows we define \mathcal{G}_0 to be the subfamily of open subsets $U \subset \mathbb{X}$ such that $\eta(\partial U) = 0$; and then define

$$\mathcal{G} = \{U \in \mathcal{G}_0 : \eta(\partial f^{-1}(U)) = 0\}.$$

Lemma 3.5. *If \mathbb{X} is a perfect set (every point is an accumulating point), then \mathcal{G}_0 generates the topology of \mathbb{X} (and also the Borel sets). Furthermore, if \mathbb{X} is a perfect set and f is continuous, then \mathcal{G} generates the topology of \mathbb{X} .*

Proof. It is enough to show that for any given $x \in \mathbb{X}$ and any $r > 0$, with $\partial B_r(x) \neq \emptyset$, there is a sequence $r_n \nearrow r$ such that $\eta(\partial(B_{r_n}(x))) = 0$.

We consider $T_{r,\varepsilon} = \{s \in (r - \varepsilon, r) : \eta(\partial B_s(x)) > 0\}$ for any fixed $0 < \varepsilon < r$. Then $\mathcal{K}_\varepsilon := \{\partial B_s(x) : s \in T_{r,\varepsilon}\}$ is a pairwise disjoint collection of compact subsets of \mathbb{X} with positive measure. From Corollary 3.3 this collection must be countable. Hence there exists $s \in (r - \varepsilon, r)$ such that $\eta(\partial B_s(x)) = 0$. Since ε, r can be taken arbitrarily close to zero, this proves what we need.

For a continuous f we replace $B_s(x)$ by $f^{-1}(B_s(x))$ in the definition of $T_{r,\varepsilon}$ and note that $\partial(f^{-1}(B_s(x)))$ is compact since $f^{-1}(B_s(x))$ is open. The same argument above applies in this case and completes the proof. \square

Lemma 3.6. *If η is a probability, then $\eta(A) = \tau(A)$ for each $A \in \mathcal{G}_0$.*

Proof. We assume that η is a probability. As $A \in \mathcal{G}_0 \iff \mathbb{X} \setminus \bar{A} \in \mathcal{G}_0$, it follows from lemma 3.4 that $\tau(\mathbb{X} \setminus A) = \tau(\overline{\mathbb{X} \setminus A}) \leq \eta(\mathbb{X} \setminus \bar{A}) \leq \eta(\mathbb{X} \setminus A)$ for every $A \in \mathcal{G}_0$. Hence $\tau(A) = \tau(\mathbb{X} \setminus (\mathbb{X} \setminus A)) \leq \eta(\mathbb{X} \setminus (\mathbb{X} \setminus A)) = \eta(A)$.

Now we assume, by contradiction, that there is some $A \in \mathcal{G}_0$ such that $\tau(A) < \eta(A)$. As $\tau(\mathbb{X} \setminus A) \leq \eta(\mathbb{X} \setminus A)$, we get $1 = \tau(\mathbb{X}) \leq \tau(A) + \tau(\mathbb{X} \setminus A) < \eta(A) + \eta(\mathbb{X} \setminus A) = \eta(\mathbb{X}) = 1$, a contradiction, completing the proof. \square

Corollary 3.7. *If $\mathcal{U} \subset \mathcal{G}_0$ is a basis of the topology, then*

$$\eta_x(\mathbb{X}) = 1 \iff \eta_x(A) = \tau_x(A), \quad \forall A \in \mathcal{U}.$$

Corollary 3.8. *If η is a probability, then $\tau(K) \leq \eta(K)$ for every closed set $K \subset \mathbb{X}$.*

Proof. Choose $\varepsilon_n \rightarrow 0$ so that $\eta(\partial B_{\varepsilon_n}(K)) = 0$ for each $n \in \mathbb{N}$. Thus $\tau(K) \leq \tau(B_{\varepsilon_n}(K)) = \eta(B_{\varepsilon_n}(K)) \rightarrow \eta(K)$. \square

Given $(\xi_j)_{j \in \mathbb{N}}$ as before and $\varphi : \mathbb{X} \rightarrow \mathbb{R}$ a bounded measurable function, we define

$$\tau(\varphi) := \limsup_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \xi_j(\varphi) = \limsup_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \int \varphi d\xi_j.$$

Clearly $\tau(\varphi + \psi) \leq \tau(\varphi) + \tau(\psi)$ and so, $\sum_{j=1}^k \tau(\phi_n) \geq \tau(\sum_{j=1}^k \phi_n)$ for any finite collection of functions $\varphi, \psi, \phi_1, \dots, \phi_k : \mathbb{X} \rightarrow \mathbb{R}$. We need the following result.

Lemma 3.9. *Let μ be a finite regular measure. If $\varphi : \mathbb{X} \rightarrow \mathbb{R}$ is a continuous function, then there exists a sequence of simple function $\varphi_n = \sum_{j=1}^{\ell_n} a_{j,n} \chi_{A_{j,n}}$ such that $A_{j,n} \in \mathcal{G}_0$, the $(A_{j,n})_j$ are pairwise disjoint for each n , $\varphi_n(x) \searrow \varphi$ for μ -almost all $x \in \mathbb{X}$ and also $\int \varphi_n d\mu \searrow \int \varphi d\mu$.*

Proof. See any standard textbook on measure and integration, e.g. [Rudin]. \square

Lemma 3.10. *If η is a probability, then $\int \varphi d\eta = \tau(\varphi)$ for each continuous function $\varphi : \mathbb{X} \rightarrow \mathbb{R}$.*

In particular, we obtain that $\tau = \eta$ is a Borel probability measure.

Proof. First we show that $\int \varphi d\eta \geq \tau(\varphi)$ for every continuous function φ .

Indeed, if $\varphi : \mathbb{X} \rightarrow \mathbb{R}$ is a continuous function, let $\varphi_n = \sum_j a_{j,n} \chi_{A_{j,n}}$ be given by Lemma 3.9. Let $a = \inf \varphi$, $\psi = \varphi - a$ and $\psi_n = \sum_j (a_{j,n} - a) \chi_{A_{j,n}}$. Note that ψ, ψ_n and $(a_{j,n} - a) \chi_{A_{j,n}}$ are nonnegative functions and $\psi_n \searrow \psi$. Since $A_{j,n} \in \mathcal{G}_0$ we get $\eta(A_{j,n}) = \tau(A_{j,n}) \forall j, n$. Thus, using that $(a_{j,n} - a) \geq 0$ we have

$$\begin{aligned} \int \psi_n d\eta &= \sum_{j=1}^{\ell_n} (a_{j,n} - a) \eta(A_{j,n}) = \sum_{j=1}^{\ell_n} (a_{j,n} - a) \tau(A_{j,n}) \\ &\geq \tau \left(\sum_{j=1}^{\ell_n} (a_{j,n} - a) \chi_{A_{j,n}} \right) = \tau(\psi_n). \end{aligned}$$

As $\psi_n \geq \psi$ implies that $\tau(\psi_n) \geq \tau(\psi)$ and since $\int \psi_n d\eta \rightarrow \int \psi d\eta$, we get

$$\int \varphi d\eta - a = \int \psi d\eta \geq \tau(\psi) = \tau(\varphi) - a.$$

That is, $\int \varphi d\eta \geq \tau(\varphi)$ for every continuous function φ .

Now, suppose that $\int \varphi d\eta > \tau(\varphi)$ for some continuous function φ . As $1 - \varphi$ is also a continuous function $\int (1 - \varphi) d\eta \geq \tau(1 - \varphi)$. Therefore

$$\begin{aligned} 1 &= \int 1 d\eta = \int ((1 - \varphi) + \varphi) d\eta = \int (1 - \varphi) d\eta + \int \varphi d\eta \\ &> \tau(1 - \varphi) + \tau(\varphi) \geq \tau((1 - \varphi) + \varphi) = \tau(1) = 1. \end{aligned}$$

This contradiction completes the proof. \square

We note that up to this point we have not used any invariance relation for the measures τ or η .

3.1. The invariant case. Now we assume that $\tau(f^{-1}(A)) = \tau(A)$ for all Borel subsets A of \mathbb{X} . This depends on the choice of the sequence $(\xi_i)_{i \in \mathbb{N}}$ and must be checked for each specific case.

Lemma 3.11. *If η is a probability measure, then η is f -invariant.*

Proof. By assumption, we have both $\tau \circ f^{-1} = \tau$ and $\eta = \tau$ on the Borel σ -algebra, thus η is f -invariant. \square

Lemma 3.12. *Let W be a compact set. Given $\delta, \varepsilon > 0$ there are a finite collection of disjoint open sets $A_1, \dots, A_s \in \mathcal{G}_0$ with $\text{diam}(A_i) < \delta \forall i$ and such that $\bigcup_{i=1}^s \overline{A_i} \supset W$ and $\sum_{i=1}^s \tau(A_i) \geq \eta(W) - \varepsilon$.*

Corollary 3.13. *The following properties are equivalent:*

- (1) η is a probability;
- (2) $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \xi_j(\varphi) = \int \varphi d\eta$ for each continuous function φ ;
- (3) $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \xi_j(\varphi)$ exists for each continuous function φ .

Proof. First we assume that η is a probability and let $\varphi \in C(\mathbb{X}, \mathbb{R})$. By Lemma 3.10 applied to $-\varphi$ we get

$$\begin{aligned} \int \varphi d\eta &= -\left(\int -\varphi d\eta \right) = -\left(\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \xi_j(-\varphi) \right) \\ &= -\left(-\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \xi_j(\varphi) \right) = \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \xi_j(\varphi). \end{aligned}$$

On the other hand, applying Lemma 3.10 to φ , we obtain

$$\int \varphi d\eta = \tau(\varphi) = \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \xi_j(\varphi).$$

This is enough to see that item (2) is true if item (1) is true. Clearly (2) implies (3).

Now we assume that (3) is true. Suppose that $\eta(\mathbb{X}) > 1$. Let $r > 0$ be small enough so that $\nu_r(\mathbb{X}) > 1$. Let $\mathcal{P} = \{P_1, \dots, P_s\} \subset \mathcal{G}_0$ be a finite collection of disjoint open sets of \mathcal{G}_0 with diameters smaller than $r/2$ and such that $\mathbb{X} = \bigcup_{j=1}^s \overline{P_j}$. Note that $\mathcal{P}_\varepsilon = \{B_{\varepsilon/2}(P_1), \dots, B_{\varepsilon/2}(P_s)\}$ is a cover of \mathbb{X} by open sets with diameter smaller than $\frac{r}{2} + \varepsilon$. As $\eta(\partial P_j) = 0 \ \forall j$, it follows from Lemma 3.2 that $\tau(P_j) \leq \tau(\overline{P_j}) \leq \tau(B_\varepsilon(P_j)) \leq \tau(P_j) + \tau(B_\varepsilon(\partial P_j)) \searrow \tau(P_j)$. That is,

$$\tau(P_j) = \tau(\overline{P_j}) = \lim_{\varepsilon \rightarrow 0} \tau(B_\varepsilon(P_j)), \ \forall j.$$

Taking only $0 < \varepsilon < r/2$, we get

$$\sum_{j=1}^s \tau(B_\varepsilon(P_j)) \geq \nu_r(\mathbb{X}) > 1.$$

Therefore, when $\varepsilon \rightarrow 0$ we get

$$\sum_{j=1}^s \tau(P_j) \geq \nu_r(\mathbb{X}) > 1.$$

As we are assuming that $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \xi_j(\varphi)$ exists for each continuous function φ and as $0 \leq \xi_j \leq 1$, it is not difficult to show that $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \xi_j(\chi_A)$ exists $\forall A \in \mathcal{G}_0$. That is,

$$\tau(A) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \xi_j(A), \ \forall A \in \mathcal{G}_0.$$

Thus,

$$1 \geq \tau(\cup_{j=1}^s P_j) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \xi_j(\cup_{j=1}^s P_j) = \sum_{j=1}^s \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \xi_j(P_j) = \sum_{j=1}^s \tau(P_j) > 1,$$

an absurd. So, η is a probability measure (recall that $\eta(\mathbb{X}) \geq 1$ always) and the proof is complete. \square

4. MEASURE ASSOCIATED TO AN ORBIT AND ERGODIC DECOMPOSITION

We recall the setting of Theorem A. Given $x_0 \in \mathbb{X}$, set $x_j = f^j(x_0)$ and define the sequence of measures $\xi_j := \delta_{x_j}$ for $j \geq 0$. Then τ satisfies $\tau(f^{-1}(A)) = \tau(A)$ for every subset A of \mathbb{X} . We denote by $\eta_x = \eta$ the measure obtained from τ with the above choices as described in Section 2. We write $\tau_x(\varphi) = \tau_x^f(\varphi) = \limsup_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^j(x) = \limsup_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \xi_j(\varphi)$ in what follows.

Remark 4.1. It is clear by definition that $\tau_x^f = \tau_{fx}^f = \tau_y^f$ for each $x, y \in \mathbb{X}$ so that $fy = x$. It follows that $\eta_x = \eta_{fx} = \eta_y$. Hence, the set \mathcal{H}_f of historic points and the set \mathcal{W}_f of wild historic points are both f -invariant: $f^{-1}(\mathcal{H}_f) = \mathcal{H}_f$ and $f^{-1}(\mathcal{W}_f) = \mathcal{W}_f$. It is also clear that $\mathcal{W}_f \supset \mathcal{H}_f$.

Proof of Theorem A. Let μ be an f -invariant finite measure.

Claim 4.2. *If μ is a f -invariant finite measure, then η_x is a probability for μ -almost every $x \in \mathbb{X}$.*

Proof of Claim 4.2. By Birkhoff ergodic theorem, we now that, given $\varphi \in C^0(\mathbb{X})$, the set \mathbb{X}_φ is the set of points $x \in \mathbb{X}$ such that $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^j(x)$ exists is of full μ measure: $\mu(\mathbb{X} \setminus \mathbb{X}_\varphi) = 0$. Hence for a dense countable collection of continuous functions $\mathcal{C} \subset C^0(\mathbb{X}, \mathbb{R})$ we have $\mu(\mathbb{X} \setminus \bigcap_{\varphi \in \mathcal{C}} \mathbb{X}_\varphi) = 0$.

If $\varphi \in C^0(\mathbb{X})$, we consider a sequence φ_k converging uniformly to φ . As

$$\left| \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^j(x) - \frac{1}{n} \sum_{j=0}^{n-1} \varphi_k \circ f^j(x) \right| \leq \|\varphi - \varphi_k\| \xrightarrow[k \rightarrow \infty]{} 0,$$

it follows that $\lim_n \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^j(x) = \lim_n \frac{1}{n} \sum_{j=0}^{n-1} \xi_j(\varphi)$ exists for every $x \in \bigcap_{\varphi \in \mathcal{C}} \mathbb{X}_\varphi$ and every $\varphi \in C^0(\mathbb{X})$. Thus, by Corollary 3.13, η_x is a probability for every $x \in \bigcap_{\varphi \in \mathcal{C}} \mathbb{X}_\varphi$. \square

Let $\mathbb{E}(\mu)$ be the set of measurable bounded functions $\varphi : \mathbb{X} \rightarrow \mathbb{R}$ such that $\int \varphi d\eta_x \geq \tau_x(\varphi)$ for μ -almost every $x \in \mathbb{X}$.

Claim 4.3. *Let $\varphi_n \in \mathbb{E}(\mu)$ be a sequence of uniformly bounded functions ($\sup |\varphi_n| \leq C \forall n$). If $\varphi(x) := \lim_{n \rightarrow \infty} \varphi_n(x)$ exists for μ -almost every $x \in \mathbb{X}$, then $\varphi \in \mathbb{E}(\mu)$.*

Proof of Claim 4.3. By the Dominated Convergence Theorem, $\int \varphi d\eta_x = \lim_n \int \varphi_n d\eta_x$. On the other hand, as $\int \varphi_n d\eta_x \geq \tau_x(\varphi_n)$ for μ -a.e. x and $\forall n \in \mathbb{N}$, we get

$$\int \varphi d\eta_x \geq \lim_n \tau_x(\varphi_n) \tag{4}$$

for μ -almost every $x \in \mathbb{X}$. As $|\tau_x(\varphi) - \tau_x(\varphi_n)| \leq \tau_x(|\varphi - \varphi_n|)$ and since (by the Ergodic Theorem) $\int_{x \in \mathbb{X}} \tau_x(|\varphi - \varphi_n|) d\mu = \int |\varphi - \varphi_n| d\mu$, we get

$$\int_{x \in \mathbb{X}} |\tau_x(\varphi) - \tau_x(\varphi_n)| d\mu \leq \int |\varphi - \varphi_n| d\mu.$$

Again by dominated convergence we obtain $\int |\varphi - \varphi_n| d\mu \rightarrow 0$. But any sequence which converges in measure has a subsequence which converges a.e., hence there exists an increasing sequence $n_k \rightarrow \infty$ such that $\lim_k \tau_x(\varphi_{n_k}) = \tau_x(\varphi)$ for μ -almost every $x \in \mathbb{X}$. Thus, by (4), we get $\int \varphi d\eta_x \geq \tau_x(\varphi)$ for μ -almost every point. \square

Claim 4.4. *For every measurable set $A \subset \mathbb{X}$ there is some $\mathbb{X}_A \subset \mathbb{X}$, with $\mu(\mathbb{X} \setminus \mathbb{X}_A) = 0$, such that $\eta_x(A) = \tau_x(A)$ for μ almost all $x \in \mathbb{X}_A$.*

Proof of Claim 4.4. First let us show that, given a measurable set $A \subset \mathbb{X}$, there is some \mathbb{X}_A^0 with $\mu(\mathbb{X} \setminus \mathbb{X}_A^0) = 0$ and satisfying $\eta_x(A) \geq \tau_x(A)$ for every $x \in \mathbb{X}_A^0$.

For this, let $K_1 \subset K_2 \subset K_3 \subset \dots \subset A$ be a nested sequence of compact sets such that $A = \bigcup_n K_n$. By Corollary 3.8,

$$\int \chi_{K_n} d\eta_x = \eta_x(K_n) \geq \tau_x(K_n) = \tau_x(\chi_{K_n}), \quad \mu - \text{almost everywhere},$$

that is, $\chi_{K_n} \in \mathbb{E}(\mu)$ for all $n \geq 1$. Therefore, as $\lim_n \chi_{K_n} = \chi_A$ and $0 \leq \chi_{K_n} \leq 1$ for all n , it follows from Claim 4.3 that there exists some set \mathbb{X}_A^0 with $\mu(\mathbb{X} \setminus \mathbb{X}_A^0) = 0$ whose points x satisfy $\eta_x(A) = \int \chi_A d\eta_x \geq \tau_x(\chi_A) = \tau_x(A)$.

To conclude the proof of the claim, given $A \subset \mathbb{X}$, set $\mathbb{X}_A = \mathbb{X}_A^0 \cap \mathbb{X}_{\mathbb{X} \setminus A}^0 \cap \mathbb{X}^0$, where $\mathbb{X}^0 = \{x \in \mathbb{X} : \eta_x(\mathbb{X}) = 1\}$. Suppose that $\eta_x(A) > \tau_x(A)$ for some $x \in \mathbb{X}_A$. In this case, $1 = \eta_x(\mathbb{X}) = \eta_x(A) + \eta_x(\mathbb{X} \setminus A) > \tau_x(A) + \tau_x(\mathbb{X} \setminus A) \geq \tau_x(A \cup (\mathbb{X} \setminus A)) = \tau_x(\mathbb{X}) = 1$ and we arrive at a contradiction. Thus $\eta_x(A) = \tau_x(A)$ for all $x \in \mathbb{X}_A$. \square

Claim 4.5. *η_x is an f -invariant probability for μ -almost every $x \in \mathbb{X}$.*

Proof. As we already know that η_x is a probability for μ -almost every x , we only need to check the invariance. For this, let $\mathcal{A} = \{A_1, A_2, A_3, \dots\}$ be a countable base of open sets for the topology of \mathbb{X} . By the last claim

$$\mu\left(\mathbb{X} \setminus \bigcap_n (\mathbb{X}_{A_n} \cap \mathbb{X}_{f^{-1}(A_n)})\right) = 0.$$

Note that, for every $x \in \bigcap_n (\mathbb{X}_{A_n} \cap \mathbb{X}_{f^{-1}(A_n)})$ and $A \in \mathcal{A}$ we have $\eta_x(A) = \tau_x(A) = \tau_x(f^{-1}(A)) = \eta_x(f^{-1}(A))$ and this implies the f -invariance of η_x . \square

It follows from Claim 4.5 that $\eta_\mu := \int_{x \in \mathbb{X}} \eta_x d\mu$ is a f -invariant finite measure. Indeed, $\eta_\mu \circ f^{-1} = \int_{x \in \mathbb{X}} \eta_x \circ f^{-1} d\mu = \int_{x \in \mathbb{X}} \eta_x d\mu = \eta_\mu$. Moreover, by Birkhoff Ergodic Theorem and Claim 4.4, we get for every bounded measurable function φ

$$\int \varphi d\mu = \int \tau_x(\varphi) d\mu = \int_{x \in \mathbb{X}} \left(\int \varphi d\eta_x \right) d\mu = \int \varphi d\eta_\mu.$$

So $\mu = \eta_\mu$ and, as a consequence, the above identities are also true for $\varphi \in L^1(\mu)$ by Dominated Convergence.

Claim 4.6. η_x is ergodic for μ almost all $x \in \mathbb{X}$.

Proof of Claim 4.6. First, since we can see $\eta_x : L^1(\mu) \rightarrow \mathbb{R}$ as linear functionals, we can also write

$$\begin{aligned} \int_{y \in \mathbb{X}} (\eta_x - \eta_y)^2 d\eta_x &= \int_{y \in \mathbb{X}} \left((\eta_x)^2 - 2\eta_x\eta_y + (\eta_y)^2 \right) d\eta_x \\ &= (\eta_x)^2 \int_{y \in \mathbb{X}} d\eta_x - 2\eta_x \int_{y \in \mathbb{X}} \eta_y d\eta_x + \int_{y \in \mathbb{X}} (\eta_y)^2 d\eta_x \\ &= (\eta_x)^2 - 2\eta_x\eta_x + \int_{y \in \mathbb{X}} (\eta_y)^2 d\eta_x = \left(\int_{y \in \mathbb{X}} (\eta_y)^2 d\eta_x \right) - (\eta_x)^2. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{x \in \mathbb{X}} \int_{y \in \mathbb{X}} (\eta_x - \eta_y)^2 d\eta_x d\mu &= \int_{x \in \mathbb{X}} \left[\left(\int_{y \in \mathbb{X}} (\eta_y)^2 d\eta_x \right) - (\eta_x)^2 \right] d\mu \\ &= \int_{x \in \mathbb{X}} \left(\int_{y \in \mathbb{X}} (\eta_y)^2 d\eta_x \right) d\mu - \int_{x \in \mathbb{X}} (\eta_x)^2 d\mu. \end{aligned}$$

As we already know that $\int_{x \in \mathbb{X}} (\eta_x)^2 d\mu = \int_{z \in \mathbb{X}} \left(\int (\eta_x)^2 d\eta_z \right) d\mu$, we conclude that

$$\int_{x \in \mathbb{X}} \int_{y \in \mathbb{X}} (\eta_x - \eta_y)^2 d\eta_x d\mu = 0.$$

As a consequence, for μ almost all $x \in \mathbb{X}$ we have $\eta_x = \eta_y$ for η_x -almost all $y \in \mathbb{X}$.

In particular, η_y is a probability for η_x -almost every y . Thus, using Corollary 3.13 and Birkhoff Ergodic Theorem, we obtain $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^j(y) = \int \varphi d\eta_y = \int \varphi d\eta_x$ for η_x -almost all $y \in \mathbb{X}$ and every $\varphi \in C^0(\mathbb{X})$. As this implies the ergodicity of η_x , which completes the proof of the claim. \square

We have obtained all the statements of the Ergodic Decomposition Theorem A. The proof is complete. \square

5. WILD HISTORIC BEHAVIOR

Here we present a proof of Theorem B.

5.1. Topological Markov Chain. We start with a countable subshift given by a denumerable set S of symbols (the alphabet, which may be finite with at least two symbols), an incidence matrix $A = (a_{i,j})_{i,j \in S}$ with 0 or 1 entries and the set $\mathbb{X} = \{\underline{x} = (x_i)_{i \geq 0} : a_{x_i, x_{i+1}} = 1, i \geq 0\}$ of admissible sequences. We assume that A is aperiodic: there exists $N \in \mathbb{Z}^+$ such that for any pair $b, c \in S$ there are $x_1, \dots, x_{N-1} \in S$ satisfying $a_{b, x_1} = a_{x_1, x_2} = \dots = a_{x_{N-2}, x_{N-1}} = a_{x_{N-1}, c} = 1$. This is the same as requiring that the denumerable infinite matrix A^N have all entries $a_{i,j}^N$ positive.

5.1.1. *Construction of wild historic orbit.* We consider the usual left shift map $\sigma : \mathbb{X} \circlearrowleft$ and then (\mathbb{X}, σ) is a mixing topological Markov chain with denumerable set of states (or symbols). The aperiodic condition on A ensures, in particular, that there exists a dense orbit and a denumerable dense subset $\text{Per}(\sigma)$ of periodic orbits; see e.g. [16, 36]. More precisely, for any given admissible sequence a_0, a_1, \dots, a_k of symbols and $q > k + N$ there exists a periodic orbit $\underline{p} = (x_k)_{k \geq 0}$ such that $x_0 = a_0, \dots, x_k = a_k$ and its (minimum) period $\pi(\underline{p})$ is q . In what follows, we write $\{\underline{p}_n\}_{n \geq 0}$ for an enumeration of a choice of one point of every periodic orbit in $\text{Per}(\sigma)$.

We consider the following integer sequence defined by recurrence

$$\ell_1 = N \quad \text{and} \quad \ell_{n+1} = 10^n \cdot \sum_{i=1}^n \ell_i, \quad n > 1.$$

We note that $\ell_{n+1} > \ell_n + N$ for each $n \geq 1$. We recall the notion of a cylinder set in \mathbb{X} : for a given sequence $\underline{a} = (a_i)_{i \geq 1} \in \mathbb{X}$ and a given positive integer n we define

$$[\underline{a}]_n = \{x = (x_i)_{i \geq 1} \in \mathbb{X} : x_1 = a_1, \dots, x_n = a_n\}.$$

Now we consider the integer sequence $\kappa_n, n \geq 1$:

$$1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, 1, 2, 3, 4, 5, 6, \dots$$

which assumes the value of any given positive integer infinitely many times and may be defined through the sequence of indexes

$$\alpha_0 = 0 \quad \text{and} \quad \alpha_{n+1} = \alpha_n + n + 1, n \geq 0$$

together with the rule $\kappa_{\alpha_n+j} = j$ for all $n \geq 1$ and $j = 1, \dots, n$. Then we join several strings of digits of \underline{p}_n to form a sequence $\underline{z} \in \mathbb{X}$ satisfying

$$\sigma^{\ell_n}(\underline{z}) \in [\underline{p}_{\kappa_n}]_{\ell_n}, \quad n \geq 1.$$

Such sequence \underline{z} is admissible (that is, it does belong to \mathbb{X}) since $\ell_{n+1} - \ell_n > N$ and so the required transitions from the position $\ell_n + \kappa_n$ to the position ℓ_{n+1} are allowed.

This sequence \underline{z} is such that its positive σ -orbit visits a very small ℓ_n -cylinder around the κ_n th element of \underline{p}_n . By the definition of κ_n and of ℓ_n , for any fixed \underline{p}_h and any given cylinder $[\underline{p}_h]_m$, with $m \geq 1$, we have for all $n \geq 1$ such that $\kappa_n = h$

(P1) $\sigma^i(\sigma^{\ell_n}(\underline{z}))$ belongs to $[\underline{p}_h]_m$ for $i = 0, \pi_h, 2\pi_h, \dots, \tau\pi_h$, where $\tau = \lfloor \ell_n / \pi_h \rfloor$ and $\pi_h = \pi(\underline{p}_h)$ is the (minimum) period of \underline{p}_h ;

(P2) $\sum_{j=\ell_n}^{2\ell_n} \chi_{[\underline{p}_h]_m}(\sigma^j(\underline{z})) \geq \tau \geq \ell_n \frac{\tau}{\tau\pi_h + r} \geq \frac{\ell_n}{\pi_h} \frac{\tau}{1+\tau} \geq \frac{\ell_n}{2\pi_h}$.

Since

$$\frac{1}{2\ell_n + 1} \sum_{j=0}^{2\ell_n} \chi_{[\underline{p}_h]_m}(\sigma^j(\underline{z})) \geq \frac{1}{2\ell_n + 1} \sum_{j=\ell_n}^{2\ell_n} \chi_{[\underline{p}_h]_m}(\sigma^j(\underline{z})) \geq \frac{\ell_n}{2\ell_n + 1} \cdot \frac{1}{2\pi_h}$$

and $\kappa_n = h$ for infinitely many positive integers n , we obtain

$$\tau_z([p_h]_m) = \limsup_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \chi_{[p_h]_m}(\sigma^j(\underline{z})) \geq \frac{1}{4\pi_h}, \quad \text{for each } h, m \geq 1. \quad (5)$$

Moreover, if $\underline{p}_h = (a_i)_{i \geq 0}$, then there are periodic orbits $p_j \in \text{Per}(\sigma)$ whose first $m+1$ symbols are a_0, \dots, a_m and whose period is any positive integer $q_j > m + jN$; and these periodic orbits also belong to $[p_h]_m$ (as well as some of its iterates, as in item (1) above whenever $\kappa_n = j$). In addition, fixing q_j , there are at least as many periodic orbits p_j with period q_j as above as letters in S^j . Therefore, (5) together with Lemma 3.4 implies

$$\eta_z([p_h]_m) \geq \eta_z \left(\bigcup_{\substack{p_j \in \text{Per}_{q_j}(\sigma) \cap [p_h]_m}} [p_j]_m \right) \geq (\#S)^j \frac{1}{m + jN}. \quad (6)$$

Even if the set of symbols is finite $\#S < \infty$, then $(\#S)^j$ increases exponentially fast in j and we get $\eta_z([p_h]_m) = \infty$. Clearly, we can use any $m \geq 1$ and also replace \underline{p}_h by $\sigma^i(\underline{p}_h)$ in the above argument for all $i \geq 1$. In addition, the set of accumulation points of the σ -orbit of \underline{z} is \mathbb{X} , because $\text{Per}(\sigma)$ is dense in \mathbb{X} .

Thus, we see that $\eta_z(A) = +\infty$ for every open subset A of \mathbb{X} , and z is a wild historic point.

Remark 5.1. The previous argument show that, in a system with dense countable subset of periodic orbits, *if a point z satisfies $\tau_z U \geq c(p) > 0$ for every neighborhood U of a periodic point p , where $c(p)$ depends on $\mathcal{O}(p)$ only, then z is a wild historic point.*

5.1.2. Genericity of wild historic points. We claim that such points form a generic subset of \mathbb{X} . Indeed, let us consider, for any given n , the integer m_n such that, for all $1 \leq j, k < n$ with $k \neq j$ we have $B(\underline{p}_j, m_n) \cap B(\underline{p}_k, m_n) = \emptyset$, where

$$B(\underline{p}_h, m_n) = \bigcup_{1 \leq i \leq \pi(\underline{p}_h)} [\sigma^i(\underline{p}_h)]_{m_n}$$

is a neighborhood of the orbit of \underline{p}_h for any $h \geq 1$. That is, we take a pairwise disjoint neighborhood of the orbit of each one of $\underline{p}_1, \dots, \underline{p}_{n-1}$, for each $n \geq 1$. We observe that clearly $m_n \nearrow +\infty$.

We then consider the family of sets

$$U_n = \left\{ \underline{x} \in \mathbb{X} \mid \forall 1 \leq j < n \exists k = k(j, n) > n : \frac{1}{k} \sum_{i=0}^{k-1} \chi_{B(\underline{p}_j, m_n)}(\sigma^i(\underline{x})) \geq \frac{1}{\pi(\underline{p}_j)} - \frac{1}{n} \right\}$$

for all integers $n \geq 1$. We note that each U_n is open in \mathbb{X} , since cylinders are simultaneously closed and open sets and σ is continuous. In addition, $\sigma^i(\underline{z}) \in U_n$ for all $n, i \geq 1$, which shows that each U_n is also dense. Thus

$$Y = \bigcap_{n \geq 1} U_n \quad (7)$$

is a generic subset of \mathbb{X} . We now show that each element \underline{w} of Y is a wild historic point: $\underline{w} \in \mathcal{W}_\sigma$.

Fix $\underline{w} \in Y$, a point $\underline{a} \in \mathbb{X}$ and $m \geq 1$, and note that

$$\tau_{\underline{w}}([\underline{a}]_m) = \limsup_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \chi_{[\underline{a}]_m}(\sigma^j(\underline{w})) \geq \frac{1}{\pi(\underline{p})}$$

where \underline{p} is a periodic point $\underline{p} \in \text{Per}(\sigma) \cap [\underline{a}]_m$ of σ in the open set $[\underline{a}]_m$. But, as before, we know that inside $[\underline{a}]_m$ we can find periodic points of σ with all periods $q > m + N$. Hence, as in (6) we obtain $\eta_{\underline{w}}([\underline{a}]_m) = \infty$. Since $\underline{a} \in \mathbb{X}$ and $m \geq 1$ were arbitrarily chosen, we conclude that $\underline{w} \in \mathcal{W}_\sigma$.

This completes the proof of the item (1) of Theorem B and shows, in particular, that any subshift of finite type (i.e., the same as \mathbb{X} but with a finite alphabet) has a generic subset of wild historic points.

Moreover, it is clear that the same construction applies verbatim to two-sided topological Markov chains, for which the left shift map σ is a homeomorphism.

Remark 5.2. We note also that a similar construction can be performed in any system with specification; see [20, Chapter 21]. However, there are classes of systems with no specification where our construction can be performed: see Example 5 and Remark 5.12 in Subsection 5.7.

Remark 5.3. By the above construction of Y , every $y \in Y$ satisfies $\tau_y U = 1/\pi(p)$ for every neighborhood U of each periodic orbit p in \mathbb{X} . Hence, from Remark 5.1, we have that *generically* $z \in \mathbb{X}$ is a wild historic point if, and only if, z satisfies $\tau_z U \geq c(p) > 0$ for every neighborhood U of a periodic point p , where $c(p)$ depends on $\mathcal{O}(p)$ only.

5.1.3. Non-existence of time averages for open and dense family of observables. We show that *at a wild historic point time averages do not exist for an open and dense subset of continuous functions*. Moreover, as shown by Jordan, Naudot and Young in [29], *all higher order averages also fail to exist*.

We say that $\varphi \in C^0(\mathbb{X}, \mathbb{R})$ is *periodically trivial* if φ has the same time average over any periodic orbit, that is $\pi(\underline{q}) \sum_{i=0}^{\pi(\underline{q})-1} \varphi(\sigma^i \underline{p}) = \pi(\underline{p}) \sum_{i=0}^{\pi(\underline{q})-1} \varphi(\sigma^i \underline{q})$, $\forall \underline{p}, \underline{q} \in \text{Per}(\sigma)$. It is clear that the family of non periodically trivial functions is an open and dense subset of $C^0(\mathbb{X}, \mathbb{R})$ with the uniform topology.

Lemma 5.4. *Let $Y \subset \mathcal{W}_\sigma$ be defined as in (7). Given $x \in Y$ and $\varphi \in C^0(\mathbb{X}, \mathbb{R})$, then $\frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j x)$ converges when $n \rightarrow \infty$ if, and only if, φ is periodically trivial.*

Proof. Let us assume that $\tilde{\varphi}(x) = \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j x)$ exists. By Remark 5.3, given $\underline{p} \in \text{Per}(\sigma)$ and any neighborhood U we have $\tau_x U = 1$. Fixing $\varepsilon > 0$ we set U so that $|\varphi(y) - \varphi(z)| < \varepsilon$ for all $y, z \in U$. Hence we can find a sequence $n_k \nearrow \infty$ such that

$\frac{1}{n_k} \sum_{j=0}^{n_k-1} \chi_U(\sigma^j x) \geq 1 - \varepsilon$ for all $k \geq 1$ and, consequently, we get

$$\frac{1}{\pi(\underline{p})} \left(\sum_{i=0}^{\pi(\underline{p})-1} \varphi(\sigma^i \underline{p}) + \varepsilon \right) + \varepsilon \|\varphi\|_0 \geq \frac{1}{n_k} \sum_{j=0}^{n_k-1} \varphi(\sigma^j x) \geq \frac{1-\varepsilon}{\pi(\underline{p})} \left(\sum_{i=0}^{\pi(\underline{p})-1} \varphi(\sigma^i \underline{p}) - \varepsilon \right) - \varepsilon \|\varphi\|_0.$$

Since $\varepsilon > 0$ is arbitrary, we obtain $\tilde{\varphi}(x) = \frac{1}{\pi(\underline{p})} \sum_{i=0}^{\pi(\underline{p})-1} \varphi(\sigma^i \underline{p})$. Because $\underline{p} \in \text{Per}(\sigma)$ is arbitrary, we conclude that φ is periodically trivial.

Reciprocally, let $\underline{p}_1, \underline{p}_2 \in \text{Per}(\sigma)$ be periodic orbits over which φ has distinct time averages. We assume without loss of generality that

$$\alpha_1 := \frac{1}{\pi(\underline{p}_1)} \sum_{i=0}^{\pi(\underline{p}_1)-1} \varphi(\sigma^i \underline{p}_1) < \frac{1}{\pi(\underline{p}_2)} \sum_{i=0}^{\pi(\underline{p}_2)-1} \varphi(\sigma^i \underline{p}_2) =: \alpha_2.$$

Let U_i be neighborhoods of $\mathcal{O}_\sigma(\underline{p}_i)$ so that $\sup \varphi|_{U_i} - \inf \varphi|_{U_i} < \varepsilon$, $i = 1, 2$ and $U_1 \cap U_2 = \emptyset$. Since $\tau_x U_i = 1$, we can find sequences $n_k(i) \nearrow \infty$ such that $\frac{1}{n_k(i)} \sum_{j=0}^{n_k(i)-1} \chi_{U_i}(\sigma^j x) \geq 1 - \varepsilon$ for all $k \geq 1$ and $i = 1, 2$. Consequently we get

$$\begin{aligned} \frac{1}{n_k(1)} \sum_{j=0}^{n_k(1)-1} \varphi(\sigma^j x) &\leq (1 - \varepsilon)(\alpha_1 + \varepsilon) + \varepsilon \|\varphi\|_0 =: a_1 \quad \text{and} \\ \frac{1}{n_k(2)} \sum_{j=0}^{n_k(2)-1} \varphi(\sigma^j x) &\geq (1 - \varepsilon)(\alpha_2 - \varepsilon) - \varepsilon \|\varphi\|_0 =: a_2. \end{aligned}$$

Moreover, $a_1 < a_2$ if, and only if, $\frac{2\varepsilon}{1-\varepsilon}(\|\varphi\|_0 + 1 - \varepsilon) < \alpha_2 - \alpha_1$ which is true for all small enough $\varepsilon > 0$. Hence $\tilde{\varphi}(x)$ does not exist. \square

5.1.4. Wild historic points are points with maximal oscillation. For any $x \in \mathbb{X}$ we follow [20] and denote by $V(x)$ the set of all weak* accumulation points of $n^{-1} \sum_{j=0}^{n-1} \delta_{\sigma^j x}$. We write $\mathbb{P}_\sigma(\mathbb{X})$ for the family of all σ -invariant Borel probability measures on \mathbb{X} endowed with the weak* topology. Since (\mathbb{X}, σ) has the specification property, we are in the setting of [20, Propositions 21.8 & 21.18].

Proposition 5.5. [20, Proposition 21.8] *The set of measures concentrated on periodic orbits is dense in $\mathbb{P}_\sigma(\mathbb{X})$.*

Proposition 5.6. [20, Proposition 21.18] *The set of points with maximal oscillation, that is, those x for which $V(x) = \mathbb{P}_\sigma(\mathbb{X})$, form a generic subset of \mathbb{X} .*

We now show that

Lemma 5.7. *Every point in the generic subset $Y \subset \mathcal{W}_\sigma$ is a point with maximal oscillation. Reciprocally, every point with maximal oscillation is a wild historic point.*

Proof. If $x \in \mathbb{X}$ is a point with maximal oscillation, then given any periodic point $\underline{p} \in \text{Per}(\sigma)$ we can find a sequence $n_k \nearrow \infty$ so that $\frac{1}{n_k} \sum_{j=0}^{n_k-1} \delta_{\sigma^j x} \xrightarrow[k \rightarrow \infty]{w^*} \frac{1}{\pi(\underline{p})} \sum_{j=0}^{\pi(\underline{p})-1} \delta_{\sigma^j \underline{p}}$. In particular,

we get that $\tau_x U = 1$ for every neighborhood U of $\{\underline{p}, \sigma \underline{p}, \dots, \sigma^{\pi(\underline{p})-1} \underline{p}\}$ and, for every small enough neighborhood V of \underline{p} , we get $\tau_x V = \pi(\underline{p})^{-1}$. Moreover, V contains distinct periodic points with all periods larger than $\pi(\underline{p}) + \ell$ for some $\ell \in \mathbb{Z}^+$. Hence $\eta_x V \geq \sum_{k \geq \pi(\underline{p}) + \ell} \frac{1}{k} = \infty$. Since periodic points are dense in \mathbb{X} , this shows that $x \in \mathcal{W}_\sigma$.

If $x \in Y$, then we have $\tau_x U = 1$ for every neighborhood U of $\mathcal{O}_\sigma(\underline{p}) = \{\underline{p}, \sigma \underline{p}, \dots, \sigma^{\pi(\underline{p})-1} \underline{p}\}$ of any given periodic point $\underline{p} \in \text{Per}(\sigma)$, by Remark 5.3. Moreover, for every small enough neighborhood V of \underline{p} , we get $\tau_x V = \pi(\underline{p})^{-1}$. Hence, given a nested fundamental family $(U_k)_{k \geq 1}$ of neighborhoods of $\mathcal{O}_\sigma(\underline{p})$ and $(V_k)_{k \geq 1}$ of \underline{p} we can find $n_k \nearrow \infty$ so that

$$\frac{1}{n_k} \sum_{j=0}^{n_k-1} \delta_{\sigma^j x} U_k \geq 1 - \frac{1}{k}, \quad \text{and} \quad \frac{1}{n_k} \sum_{j=0}^{n_k-1} \delta_{\sigma^j x} V_k \geq \frac{1}{\pi(\underline{p})} - \frac{1}{k}, \quad \forall k \geq 1.$$

Thus denoting μ a weak* accumulation point of $\frac{1}{n_k} \sum_{j=0}^{n_k-1} \delta_{\sigma^j x}$, we obtain $\mu(U_k) = 1$ and $\mu(V_k) = \frac{1}{\pi(\underline{p})}$ for all $k \geq 1$, and conclude that $\mu = \frac{1}{\pi(\underline{p})} \sum_{j=0}^{\pi(\underline{p})-1} \delta_{\sigma^j \underline{p}}$.

Because $\underline{p} \in \text{Per}(\sigma)$ was arbitrary, we have shown that $V(x)$ contains all probability measures supported on periodic points. Since these periodic measures are dense in $\mathbb{P}_\sigma(\mathbb{X})$, we conclude that $V(x) = \mathbb{P}_\sigma(\mathbb{X})$ and x has maximal oscillation. \square

Example 3. The “extreme non-normal numbers” and “extreme non-normal continued fractions” studied by Olsen in [42, 39] are wild historic points, and so are generic subsets of the interval $[0, 1]$. Since Liouville numbers (see e.g. Niven [38] for a classical introduction) also form a generic subset of the real line, we have that generically all Liouville numbers are wild historic and extreme non-normal points for the maps $T_b : [0, 1] \circlearrowleft, x \mapsto bx \bmod 1$ for all positive integers $b \geq 2$.

In fact, extreme non-normal numbers are given by sequences $(a_n)_n \in \mathbb{X} := \mathbb{N}^\mathbb{N}$ so that the frequency of a sequence $b \in \mathbb{N}^h$ of length $h \in \mathbb{N}$ in the first ℓ elements

$$\Pi((a_n)_n, b, \ell) = \frac{1}{\ell} \#\{1 \leq j \leq \ell - h : a_j = b_1, a_{j+1} = b_2, \dots, a_{j+h-1} = b_h\}$$

accumulates, when $\ell \rightarrow \infty$, on all the possible frequency vectors $p \in S_h \subset \Delta_h = \{(p_b)_{b \in \mathbb{N}^h} : \sum_{b \in \mathbb{N}^h} p_b = 1\}$ of sequences of h symbols, for each $h \geq 1$; cf. the notion of points with maximal oscillation.

An extremely non-normal admissible sequence $\underline{x} = (x_i)_{i \geq 0} \in \mathbb{X}$ visits every given fixed cylinder $[\underline{p}_k]_h$ infinitely many times. Hence, for every $h, m \geq 1$ there exists $n \geq 1$ such that $\sigma^{\ell_m} \underline{x} \in [\underline{p}_h]_{\ell_m}$. We thus obtain (P1) and (P2) as in Subsection 5.1.1 and since this holds for infinitely many values of m , we also get (5).

The relation with number theory and the maps T_b is given by the partition $\{J_i = [(i-1)/b, i/b[: i = 1, \dots, b\}$ of the unit interval into equally sized intervals; and the connection with the Gauss map G is provided by the partition $\{L_i =]1/(i+1), 1/i], i \geq 1\}$. We associate to a sequence $\underline{a} = (a_n)_{n \geq 1} \in \mathbb{Y} = \{1, \dots, b\}^\mathbb{N}$ or $\underline{b} = (b_n)_{n \geq 1} \in \mathbb{X} = \mathbb{N}^\mathbb{N}$ the point $h(\underline{a}) = \cap_{n \geq 1} T_b^{-n} \overline{J_{a_n}}$ and $g(\underline{b}) = \cap_{n \geq 1} G^{-n} \overline{L_{b_n}}$; obtaining surjective continuous maps $h : \mathbb{Y} \rightarrow [0, 1], g : \mathbb{X} \rightarrow [0, 1]$ such that $T_b \circ h = h \circ \sigma_\mathbb{Y}$ and $G \circ g = g \circ \sigma_\mathbb{X}$. We can then

use the extremely non-normal sequences to build the extremely non-normal numbers and continued fractions; see below for instances of similar constructions in other settings.

5.2. Open continuous and transitive expansive maps. Let $f : X \rightarrow X$ be a continuous map of a compact metric space. We say that f is *positively expansive*, that is

$$\exists \delta > 0 : (d(f^n x, f^n y) \leq \delta, \forall n \geq 0) \implies x = y.$$

An apparently stronger notion is that of distance-expanding. We say that f is *distance-expanding* if there are constants $\lambda > 1, \eta > 0$ and $n \geq 0$ so that for all $x, y \in \mathbb{X}$

$$d(x, y) \leq 2\eta \implies d(f^n x, f^n y) \geq \lambda d(x, y).$$

We are then in the setting of

Theorem 5.8. [48, Theorem 4.6.1] *If a continuous map $f : X \rightarrow X$ of a compact metric space is positively expansive, then there exists a metric on X , compatible with the topology, such that f is distance-expanding with respect to this metric.*

Finally, open distance-expanding maps on compact metric spaces admit Markov partitions [48, Theorem 4.5.2] and hence the dynamics of these maps is semiconjugated to a Topological Markov Chain, as follows.

Theorem 5.9. [48, Theorems 4.3.12 & 4.5.7] *Let $f : X \rightarrow X$ be an open distance-expanding map. Then there exists a $d \times d$ matrix $A \in \{0, 1\}^{d \times d}$ such that the corresponding one-sided topological Markov Chain $\mathbb{X} = \Sigma_A$ with the left shift map $\sigma : \mathbb{X} \rightarrow \mathbb{X}$ admits a continuous surjective mapping $\pi : \mathbb{X} \rightarrow X$ such that $\pi \circ \sigma = f \circ \pi$ and a generic subset $Z \subset X$ so that $\pi|_{\pi^{-1}(Z)} : \pi^{-1}(Z) \rightarrow Z$ is injective.*

Moreover, f admits a countable dense subset of periodic points and if, in addition, f is transitive, then f is topologically mixing.

Thus π is a semiconjugation, generically a conjugation, and we have $\tau_{\pi x}^f = \pi_*(\tau_x^\sigma)$ for $x \in \mathbb{X}$, where τ_y^f and τ_x^σ represent the measures τ with respect to the f and σ dynamics, respectively. Therefore, $\pi(\mathcal{W}_\sigma \cap \pi^{-1}Z) \subset \mathcal{W}_f \cap Z$ is a generic subset of X if \mathcal{W}_σ is a generic subset of \mathbb{X} .

To conclude the proof of item (2) of Theorem B we note that, if f is an open continuous expansive and topologically transitive map, then we can find a compatible metric with respect to which f becomes topologically mixing and conjugated on a generic subset to the mixing topological Markov Chain $\sigma : \mathbb{X} \rightarrow \mathbb{X}$. In this setting, we know that \mathcal{W}_σ is a generic subset of \mathbb{X} and so f admits a generic subset \mathcal{W}_f of wild historic points.

5.3. Transitive local homeomorphisms with induced full branch Markov map.

For item (3) of Theorem B we recall that $f : X_0 \rightarrow X$ is a local homeomorphism where X_0 is an open dense subset of a compact metric space. We assume that f is topologically transitive and that there exists a open connected subset $\Delta \subset X_0$ and an induced full branch Markov map $F : G \rightarrow \Delta \supset G$. This means

- (a) there exists a function $R : G \rightarrow \mathbb{Z}^+$ where G is an open dense subset of Δ and $Fz = f^{R(z)}(z)$ for all $z \in G$;

- (b) there exists an at most denumerable partition $\mathcal{P} = \{\Delta^i\}_{i \geq 1}$ of G such that $R \mid \Delta^i \equiv R_i$ is constant on every element of \mathcal{P} ;
- (c) $F_i = F \mid_{\Delta^i} : \Delta^i \rightarrow \Delta$ is an expanding homeomorphism: there exists $\sigma > 1$ such that $d(F_i x, F_i y) \geq \sigma d(x, y)$ for all $x, y \in \Delta^i, i \geq 1$.

The assumptions on F ensure that there exist a countable dense subset of periodic orbits for F in Δ and, moreover, there exists a surjective continuous map $h : \mathbb{X} \rightarrow \Delta$ such that $h \circ \sigma = F \circ h$, where $\sigma : \mathbb{X} \rightarrow \mathbb{X}$ is the full shift with countable number of symbols. Indeed, we just have to define $h(\theta) = \bigcap_{n \geq 0} \overline{F^{-n} \Delta^{\theta_n}}$ for $\theta \in \mathbb{X}$, which is well-defined by item (c) above. Setting $Z = \Delta \setminus \left(\bigcup_{n \geq 1} F^{-1}(\Delta \setminus G) \right)$ we have a generic subset of Δ such that $h \mid_{h^{-1}Z} : h^{-1}Z \rightarrow Z$ is injective. Thus h is a conjugation between generic subsets.

We write $\tau_x^F(A) = \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{F^j x}(A)$ for all $A \subset \Delta$ and also τ_y^f, τ_z^σ for the same notions for f -orbits of $y \in X$ and σ -orbits of $z \in \mathbb{X}$, respectively. We also write \mathcal{W}_F for the set of wild historic points for $F : G \rightarrow \Delta \supset G$, \mathcal{W}_f the set of wild historic points for $f : X_0 \rightarrow X$ and \mathcal{W}_σ for the set of wild historic points of $\sigma : \mathbb{X} \rightarrow \mathbb{X}$.

It is easy to see that $\tau_{hx}^F = h_* \tau_x^\sigma$ for all $x \in \mathbb{X}$ and consequently that $h(\mathcal{W}_\sigma \cap h^{-1}Z) \subset \mathcal{W}_F \cap Z$ is a generic subset of Δ . Since $\bigcup_{n \geq 0} f^{-n} \Delta$ is dense in X by transitivity, it is enough to show that $\mathcal{W}_F \subset \mathcal{W}_f$ to conclude that \mathcal{W}_f is a generic subset of X , because \mathcal{W}_f is f -invariant.

Lemma 5.10. $\mathcal{W}_F \subset \mathcal{W}_f$.

Proof. We observe that every $p \in \text{Per}(F)$ with period $\pi_F(p)$ is such that $p \in \text{Per}(f)$ with period $\pi_f(p) = S_{\pi_F(p)}^f R(p) = \sum_{i=0}^{\pi_F(p)-1} R(F^i p)$. In addition, since $\overline{\text{Per}(F)} = \Delta$ and by transitivity $\bigcup_{n \geq 0} f^n \Delta = X$, we conclude that $\overline{\text{Per}(f)} = X$.

From Remark 5.3, there exists a topologically generic subset Y of \mathcal{W}_F and $\gamma : \text{Per}(F) \rightarrow \mathbb{R}^+$ such that for all $y \in Y$ we get $\tau_y^F(U) \geq \gamma(p)$ for each neighborhood U of $p \in \text{Per}(F)$. By the relation between the periods of $p \in \text{Per}(F)$ with respect to F and to f , we obtain $\tau_y^f(U) \geq \gamma(p) \frac{\pi_F(p)}{\pi_f(p)} = \gamma(p) \frac{\pi_F(p)}{S_{\pi_F(p)}^f R(p)} = \xi(p) > 0$.

Since $p \in \text{Per}(F)$ is also a periodic point for f and clearly $\xi(p) = \xi(fp)$, we have obtained $\xi : P \rightarrow \mathbb{R}^+$ so that, for every neighborhood V of a point q in the dense subset $P := \bigcup_{i \geq 0} f^i \text{Per}(F)$ of $\text{Per}(f)$, it holds $\tau_y^f(V) \geq \xi(q)$. This is enough to conclude, again by Remark 5.3, that $y \in Y$ belongs to \mathcal{W}_f . \square

To prove item (3) of Theorem B, just recall that \mathcal{W}_f is f -invariant. Since $\mathcal{W}_f \supset \mathcal{W}_F$ is residual in Δ (i.e., it contains a denumerable intersection of open and dense subsets of Δ) and f is a transitive local homeomorphism on an open dense subset, then $\mathcal{W}_f \supset \bigcup_{n \geq 1} f^n(\mathcal{W}_f \cap \Delta)$ is a residual subset of X .

5.4. Special semiflows over local homeomorphisms. For item (4) of Theorem B, we keep f as in the previous setting, take a measurable function $r : X \rightarrow [r_0, +\infty]$, where we fix $r_0 > 0$ and assume $r \mid_{X_0} < \infty$, and consider the special semiflow over f with roof function r , that we denote by $\phi_t : X_f^r \rightarrow X_f^r$. Here $X^r = \{(x, s) \in X \times [0, +\infty] : 0 \leq s < r(x)\}$ is the ambient space of the flow and $X \simeq X \times \{0\}$ becomes a cross-section for ϕ_t .

In this setting we analogously define a premeasure $\tau_{(x,s)}^\phi(E) = \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T 1_E(\phi_t(x, s)) dt$ and then build the measure $\eta_{(x,s)}^\phi$ from the premeasure as explained before. We wrote \mathcal{W}_ϕ for the set of wild historic points for ϕ_t .

If we assume that $r \leq r_1$ for some constant $r_1 > r_0$, then for a given $x \in \widetilde{X}_0$ the hitting times at X of the forward ϕ -orbit of $(x, 0)$ are $T_n = T_n(x) = S_n^f r(x) = \sum_{i=0}^{n-1} r(f^i x) \in [r_0 n, r_1 n]$ for all $n \geq 1$. For an elementary open set $A \times I$ of X^r , where A is an open in X and I is an open interval in \mathbb{R} so that $A \times I \subset X^r$, we have

$$\begin{aligned} \tau_{(x,0)}^\phi(A \times I) &= \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T 1_{A \times I}(\phi_t(x, 0)) dt \\ &\geq \limsup_{n \rightarrow \infty} \frac{n}{T_n} \cdot \frac{1}{n} \sum_{i=0}^{n-1} 1_A(f^i x) \cdot \text{Leb}(I) = \frac{\text{Leb}(I)}{r_1} \tau_x^f(A). \end{aligned} \quad (8)$$

This shows that $\mathcal{W}_f \subset \mathcal{W}_\phi$ and so \mathcal{W}_ϕ contains a residual subset of X^r from item (3) of Theorem B already proved.

5.5. Stable sets of Axiom A basic sets. For item (5) of Theorem B we use known results from the theory of Hyperbolic Dynamical Systems; for the setting and relevant definitions, see [53, 15, 16, 17].

5.5.1. The Axiom A diffeomorphism case. We recall that each basic set Λ of an Axiom A diffeomorphism $f : M \rightarrow M$ of a compact manifold is a finite pairwise disjoint union $\cup_{i=1}^k \Lambda_i = \cup_{i=1}^{n-1} f^i(\Lambda_1)$ of compacta such that $f^n(\Lambda_1) = \Lambda_1$, and $f^n|_{\Lambda_1}$ is semiconjugated to a subshift of finite type \mathbb{X} . That is, there exists a surjective continuous map $h : \mathbb{X} \rightarrow \Lambda_1$ such that $h \circ \sigma = f \circ h$.

This ensures that there is a generic subset $Y_1 = h(\mathcal{W}_\sigma)$ of wild historic points in Λ_1 for the action of f^n , which in turn generates a generic subset $\hat{Y} = Y_1 \cup f(Y_1) \cup \dots \cup f^{n-1}(Y_1)$ of wild historic points of Λ for the action of f . In addition, since points in the stable set of Λ

$$W^s(\Lambda) = \{z \in M : \text{dist}(f^n(z), \Lambda) \xrightarrow{n \rightarrow +\infty} 0\}$$

belong to the stable set of some point of Λ , that is, $W^s(\Lambda) = \cup_{x \in \Lambda} W^s(x)$, where

$$W^s(x) = \{z \in M : \text{dist}(f^n(z), f^n(x)) \xrightarrow{n \rightarrow +\infty} 0\},$$

and points in the same stable manifold have the same asymptotic sojourn times, we have that $W^s(\hat{Y}) = \cup_{x \in \hat{Y}} W^s(x)$ is a topologically generic subset of $W^s(\Lambda)$ formed by wild historic points, and $\mathcal{W}_f \supset Y$.

5.5.2. The Axiom A vector field case. For a basic set Λ of an Axiom A vector field X on a compact manifold, we analogously have that Λ is semiconjugated to a suspension flow $\phi_t : \mathbb{X}^r \curvearrowright$ over a two-sided subshift of finite type \mathbb{X} with a bounded roof function $r : \mathbb{X} \rightarrow [r_0, r_1]$ for some $0 < r_0 < r_1, r_0, r_1 \in \mathbb{R}$; see [15, 17].

From item (3) of Theorem B already proved, we see that \mathcal{W}_ϕ is a topologically generic subset of \mathbb{X}^r . If $h : \Lambda \rightarrow \mathbb{X}^r$ is the semiconjugation between the actions of X on Λ and ϕ_t on \mathbb{X}^r , we get $\mathcal{W}_X \cap \Lambda \supset h^{-1}\mathcal{W}_\phi$. Hence the set of wild historic points on Λ is again a topologically generic subset.

This completes the proof of item (5) of Theorem B.

5.6. Expanding measures. For item (6) of Theorem B, we use [47]. We recall that a $C^{1+\alpha}$ -map $f : M \setminus \mathcal{C} \rightarrow M$ of a compact manifold M is *non-flat* if f is a local diffeomorphism everywhere except at a *non-degenerate singular/critical set* \mathcal{C} , that is, $M \setminus \mathcal{C}$ is open and dense in M and there are $\beta, B > 0$ so that

$$(S1) \quad \frac{1}{B} d(x, \mathcal{S})^\beta \leq \frac{\|Df(x)v\|}{\|v\|} \leq B d(x, \mathcal{C})^{-\beta};$$

$$(S2) \quad \left| \log \|Df(x)^{-1}\| - \log \|Df(y)^{-1}\| \right| \leq B \frac{d(x, y)}{d(x, \mathcal{C})^\beta};$$

for every $x, y \in M \setminus \mathcal{C}$ with $d(x, y) < d(x, \mathcal{C})/2$ and $v \in T_x M \setminus \{0\}$.

An invariant *expanding measure* for f is a probability measure μ satisfying

- *non-flatness*: $\mu(\mathcal{C}) = 0$, $f_*\mu \ll \mu$, μ admits a Jacobian $J_\mu f(x)$ with respect to f well defined and positive μ -a.e. and, for μ -a.e. $x, y \in M \setminus \mathcal{C}$ with $d(x, y) < d(x, \mathcal{C})/2$, we have

$$\left| \log \frac{J_\mu f(x)}{J_\mu f(y)} \right| \leq \frac{B}{d(x, y)^\beta} \cdot d(x, y);$$

- *non-uniformly expansion*: there exists $c > 0$ such that for μ -a.e. x

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j x)^{-1}\| \leq -c;$$

- *slow recurrence* \mathcal{C} : for every $\varepsilon > 0$ there exists $\delta > 0$ such that for μ -a.e. x

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \left| \log d_\delta(f^j x, \mathcal{S}) \right| < \varepsilon;$$

Then [47, Theorem B] ensures, in particular, that every expanding invariant measure μ for a non-flat map f of a compact manifold admits an induced full branch Markov map defined on an open subset of $\text{supp } \mu$. Hence the set $\mathcal{W}_f \cap \text{supp } \mu$ is a topologically generic subset of $\text{supp } \mu$ by item (2) of Theorem B already proved. This completes the proof of item (6) of Theorem B.

Example 4. As detailed in [47] the class of expanding measures presented above encompasses

- (1) the absolutely continuous invariant probability measure for non-uniformly expanding maps introduced in [3, 2] including the Viana maps from [58];
- (2) each piecewise expanding $C^{1+\alpha}$ maps of the interval, which include the Lorenz-like transformations of the interval studied in [5];

- (3) in addition, absolutely continuous invariant probability measures for smooth multidimensional expanding maps studied in [23, 51] are also expanding measures.

5.7. Hyperbolic measures for diffeomorphisms and flows. For item (7) of Theorem B, we use the following well-known result from the theory of non-uniform hyperbolicity; see [31, Supplement] and [10] for a modern presentation of this theory. We write $\text{Per}_h(f)$ for the set of hyperbolic periodic points of the map f .

Theorem 5.11. [31, Theorem S.5.3, pp. 694-695]. *If μ is an ergodic hyperbolic continuous measure (that is, μ has no atomic part) for a $C^{1+\alpha}$ diffeomorphism $f : M \curvearrowright$ of a compact manifold, $\alpha > 0$, then there exists $p \in \text{Per}_h(f)$ such that $\text{supp}(\mu) \subset H(p) = \overline{W^s(p)} \cap \overline{W^u(p)}$, that is, the support of μ is contained in the homoclinic class of p . In particular, $\text{supp}(\mu) \subset \overline{\text{Per}_h(f)}$.*

5.7.1. The diffeomorphism case. To prove item (6) of Theorem B, let μ be an ergodic non-atomic hyperbolic probability measure for a $C^{1+\alpha}$ diffeomorphism $f : M \curvearrowright$ of a compact manifold, for some $\alpha > 0$.

The Birkhoff-Smale Theorem [52] ensures that every neighborhood of a homoclinic point intersects a horseshoe. A compact f invariant subset Λ is a *horseshoe* if there are $s, k \in \mathbb{Z}^+$ such that Λ decomposes as the disjoint union of $\Lambda_0 \cup \dots \cup \Lambda_{k-1}$ satisfying $f^k(\Lambda_i) = \Lambda_i$, $f(\Lambda_i) = \Lambda_{i+1 \bmod k}$ and $f^k|_{\Lambda_0}$ is topologically conjugated to a full shift in s symbols. This implies after Theorem 5.11 that densely in the support of μ we can find horseshoes.

From item (1) already proved, we have that densely in $\text{supp}(\mu)$ there are points with wild historic behavior in some horseshoe $\Lambda_z \subset \text{supp}(\mu)$: let $z_n \in \text{supp}(\mu)$ be an enumeration of such dense subset. This means in particular that $\eta_{z_n}(A) = +\infty$ for all open subsets A of Λ_{z_n} , for all $n \geq 1$.

For $z \in \text{supp}(\mu)$, $n \geq 1$ and $\varepsilon > 0$ let

$$B(z, n, \varepsilon) = \{x \in \text{supp}(\mu) : d(f^i x, f^i z) < \varepsilon, \forall 0 \leq i < n\}$$

be the (n, ε) -ball (dynamical ball) around z . We consider $Y_i = \bigcap_{n \geq 1} \bigcup_{k \geq n} f^{-k} B(z_i, n, 1/n)$ for each $i \in \mathbb{Z}^+$. By construction, for a given $w \in Y_i$ with $i \in \mathbb{Z}^+$ fixed, we can find k_n such that $f^{k_n} w \in B(z_i, n, 1/n)$ and $k_n > e^n$. So, for any given $\varepsilon > 0$ and $y \in \Lambda_{z_i}$ and all big enough n so that $1/n < \varepsilon$

$$\frac{1}{n + \ell} \sum_{j=0}^{n+\ell-1} 1_{B(y, 2\varepsilon)}(f^j w) \geq \frac{\ell}{n + \ell} \cdot \frac{1}{\ell} \sum_{j=0}^{\ell-1} 1_{B(y, \varepsilon)}(f^j z_i)$$

for all $0 \leq \ell < k_n$. That is $\tau_w(B(y, 2\varepsilon)) \geq \tau_{z_i}(B(y, \varepsilon))$. In particular, this means that $\eta_w|_{\Lambda_{z_i}} \geq \eta_{z_i}|_{\Lambda_{z_i}}$ and so $\eta_w(B(z_i, \varepsilon)) = +\infty$.

Now let us consider $Y = \bigcap_{n \geq 1} \bigcap_{i=1}^n \bigcup_{k \geq n} f^{-k} B(z_i, n, 1/n)$. Since μ is f -ergodic, we have that Y is a denumerable intersection of the open and dense subsets $\bigcap_{i=1}^n \bigcup_{k \geq n} f^{-k} B(z_i, n, 1/n)$ of the compact set $\text{supp}(\mu)$, thus Y is residual in $\text{supp}(\mu)$. Moreover, for $w \in Y$ and A a non-empty open subset of $\text{supp} \mu$, there exist $i \geq 1, \varepsilon > 0$ such that $B(z_i, \varepsilon) \subset A$, thus $\eta_w(A) = +\infty$, showing that every $w \in Y$ has wild historic behavior: $W_f \cap \text{supp}(\mu) \supset Y$.

This completes the proof for the support of a non-atomic ergodic hyperbolic probability measure for a C^{1+} diffeomorphism.

5.7.2. The vector field case. Let μ be a non-atomic ergodic hyperbolic probability measure for a C^{1+} vector field X of a compact manifold M . Then a version of Theorem 5.11 also holds, that is, $\text{supp } \mu$ is contained in the homoclinic class of some hyperbolic periodic point p for the flow ϕ_t generated by X .

Hence the Birkhoff-Smale Theorem guarantees that every neighborhood of a homoclinic point intersects the *suspension* of a horseshoe, by a bounded roof function. We observe that each suspension flow with bounded roof function over a horseshoe admits a topologically generic subset of wild hyperbolic points, from item (4) of Theorem B already proved.

Then the same argument as above shows that there exists a topologically generic subset of wild hyperbolic points in $\text{supp } \mu$. Indeed, according the results on existence, uniqueness and continuity of solutions of Ordinary Differential Equations, for each $x \in M$ and every $T > 0$ there exists $\varepsilon = \varepsilon(x, T) > 0$ so that the dynamical ball

$$B(x, T, \varepsilon) = \{y \in M : d(\phi_t x, \phi_t y) < \varepsilon, \forall 0 \leq t \leq T\}$$

is an open neighborhood of x . Note that $\varepsilon(x, T) \xrightarrow{T \rightarrow +\infty} 0$. Hence, from the Birkhoff-Smale Theorem, we have $D \subset \text{supp } \mu$ a denumerable dense subset of wild historic points, each point in some suspended horseshoe. Given an enumeration $\{z_i\}_{i \in \mathbb{Z}^+}$ of D , we consider the set $Y = \bigcap_{n \geq 1} \bigcup_{i=0}^n \bigcup_{k \geq n} \phi_{-k} B(z_i, n, \varepsilon(z_i, n))$ which is again topologically generic in $\text{supp } \mu$. Given $y \in \text{supp } \mu$, $x \in Y$, $z_i \in D$ and $\delta > 0$, for all large enough $n \in \mathbb{Z}^+$ there exists $k = k_n > e^n$ so that $\phi_k x \in B(z_i, n, \varepsilon(z_i, n))$ and $\delta > \varepsilon(z_i, n)$ and

$$\frac{1}{n + \ell} \int_0^n 1_{B(y, 2\delta)}(\phi_t x) dt \geq \frac{\ell}{n + \ell} \cdot \frac{1}{\ell} \int_0^\ell 1_{B(y, \delta)}(\phi_t z_i) dt$$

for each $0 \leq \ell < k_n$. This shows that $\tau_x^\phi(B(y, 2\delta)) \geq \tau_{z_i}^\phi(B(y, \delta))$ and so, since $y \in \text{supp } \mu$ and $\delta > 0$ where arbitrarily chosen, we get $\eta_x^\phi \geq \eta_{z_i}^\phi$ for all $i \in \mathbb{Z}^+$. Thus $\eta_x^\phi(A) = +\infty$ for all each subset A of $\text{supp } \mu$, since $D = \{z_i\}_{i \in \mathbb{Z}^+}$ is dense in $\text{supp } \mu$. We proved $\mathcal{W}_\phi \supset Y$.

This finishes the proof of item (7) of Theorem B and completes the proof of Theorem B.

Example 5. Every geometric Lorenz attractor, the classical Lorenz attractor, and every singular-hyperbolic attractor, admit an ergodic hyperbolic measure which is also physical and, in particular, non-atomic; see [5, 4]. Hence, the set of wild hyperbolic points inside these attractors is a topologically generic subset. In addition, the stable set for this class of attractors also admits a topologically generic set of wild historic points.

Remark 5.12. Recently it has been show by Sumi, Varandas and Yamamoto [54] that Lorenz-like attractors do not satisfy specification. Hence, the genericity of wild historic points is more general than the genericity of points with maximal oscillation in systems with specification.

6. CONTINUOUS DYNAMICAL SYSTEMS

Here we present a proof of Theorem C and Corollary D

We now assume that not only τ is f -invariant, but also that $f : \mathbb{X} \rightarrow \mathbb{X}$ is a continuous map on a metric space \mathbb{X} which is *perfect*, that is, there are no isolated points.

Theorem 6.1. *If \mathbb{X} is perfect and f is continuous, then η_x is a non trivial invariant measure for every $x \in \mathbb{X}$.*

Proof. We claim that $\eta_x(A) \leq \eta_x(f^{-1}(A))$ for all $A \in \mathcal{G}$. Let $A \in \mathcal{G}$. Let $r > 0$ and $\mathcal{C} \in \mathcal{A}(r, \bar{A})$. As \bar{A} is compact, let $\mathcal{C}' \subset \mathcal{C}$ bet a finite subcover of \bar{A} . Of course that, $\mathcal{C}' \in \mathcal{A}(r, \bar{A})$ and $\nu_r(\bar{A}) \leq \sum_{V \in \mathcal{C}'} \tau_x(V) \leq \sum_{V \in \mathcal{C}} \tau_x(V)$. Given any set V and $\varepsilon > 0$, define

$$[V]_\varepsilon = V \setminus B_\varepsilon(\partial V).$$

As \bar{A} is compact and \mathcal{C}' is finite, it is easy to see that there is some $\varepsilon_0 > 0$ such that

$$\mathcal{C}'_\varepsilon := \{[V]_\varepsilon; V \in \mathcal{C}'\} \in \mathcal{A}(r, \bar{A})$$

for every $0 < \varepsilon \leq \varepsilon_0$.

Moreover, $\nu_r(\bar{A}) \leq \sum_{V \in \mathcal{C}'_\varepsilon} \tau_x(V) \leq \sum_{V \in \mathcal{C}'} \tau_x(V) \leq \sum_{V \in \mathcal{C}} \tau_x(V)$, whenever $0 < \varepsilon \leq \varepsilon_0$.

It follows from Corollary 3.3 that we can choose $0 < \varepsilon \leq \varepsilon_0$ so that $[V]_\varepsilon$ and $f^{-1}([V]_\varepsilon) \in \mathcal{G}$ for all $V \in \mathcal{C}'$. That is, $V, f^{-1}(V) \in \mathcal{G} \ \forall V \in \mathcal{C}'_\varepsilon$.

We write $\mathcal{C}'_\varepsilon = \{V_1, \dots, V_s\}$ and set $P_1 = V_1 \cap \bar{A}$, $P_2 = (V_2 \cap \bar{A}) \setminus P_1$, \dots , $P_s = (V_s \cap \bar{A}) \setminus (P_1 \cup \dots \cup P_{s-1})$. Observe that

- (1) $P_j \cap P_k = \emptyset$ if $j \neq k$,
- (2) $B_{\varepsilon'}(P_j) \subset V_j$ for every $0 < \varepsilon' < \varepsilon$ and every $j = 1, \dots, s$,
- (3) $\bigcup_{j=1}^s P_j = \bar{A}$ and
- (4) $\eta_x(\partial P_j) = \eta_x(\partial(f^{-1}(P_j))) = 0$ for all $j = 1, \dots, s$.

Thus,

$$\nu_r(\bar{A}) \leq \sum_{j=1}^s \tau_x(B_{\varepsilon'}(P_j)) \leq \sum_{j=1}^s \tau_x(V_j).$$

On the other hand $\eta_x(\partial P_j) = 0$, we get $\lim_{\varepsilon' \rightarrow 0} \tau_x(B_{\varepsilon'}(\partial P_j)) = 0$ and so,

$$\lim_{\varepsilon' \rightarrow 0} \tau_x(B_{\varepsilon'}(P_j)) = \tau_x(P_j) \ \forall j = 1, \dots, s.$$

So, we can conclude that

$$\nu_r(\bar{A}) \leq \sum_{j=1}^s \tau_x(P_j).$$

As $\chi_{\mathbb{Y}}(f^j(x)) = \chi_{f^{-1}(\mathbb{Y})}(f^{j-1}(x))$, for every $\mathbb{Y} \subset \mathbb{X}$, we get that $\tau_x \circ f^{-1} = \tau_x$ and so,

$$\nu_r(\bar{A}) \leq \sum_{j=1}^s \tau_x(P_j) = \sum_{j=1}^s \tau_x(f^{-1}(P_j)).$$

As $\eta_x(\partial f^{-1}(P_j)) = 0$, we get $\tau_x(\partial f^{-1}(P_j)) = 0$ (from Lemma 3.2). Applying Lemma 3.4 to $\text{inter } f^{-1}(P_j)$, it follows that

$$\tau_x(f^{-1}(P_j)) \leq \eta_x(\overline{\text{inter } f^{-1}(P_j)}) \leq \eta_x(\text{inter}(f^{-1}(P_j))) \leq \eta_x(f^{-1}(P_j)).$$

Therefore,

$$\nu_r(\bar{A}) \leq \sum_{j=1}^s \tau_x(f^{-1}(P_j)) \leq \sum_{j=1}^s \eta_x(f^{-1}(P_j)) = \eta_x\left(\sum_{j=1}^s f^{-1}(P_j)\right) = \eta_x(f^{-1}(\bar{A})).$$

Taking $r \rightarrow 0$, we get

$$\eta_x(A) = \eta_x(\bar{A}) \leq \eta_x(f^{-1}(\bar{A})) = \eta_x(f^{-1}(A)),$$

whenever $A \in \mathcal{G}$.

Note that $A \in \mathcal{G} \Rightarrow \mathbb{X} \setminus \bar{A} \in \mathcal{G}$. Suppose that there are some $A \in \mathcal{G}$ such that $\eta_x(A) < \eta_x(f^{-1}(A))$. Thus $\eta_x(\bar{A}) < \eta_x(f^{-1}(\bar{A}))$ and then we reach a contradiction:

$$\begin{aligned} \eta_x(\mathbb{X}) &= \eta_x(\bar{A}) + \underbrace{\eta_x(\mathbb{X} \setminus \bar{A})}_{\leq f^{-1}(\mathbb{X} \setminus \bar{A})} < \eta_x(f^{-1}(\bar{A})) + \eta_x(f^{-1}(\mathbb{X} \setminus \bar{A})) = \eta_x(\mathbb{X}). \end{aligned}$$

As a consequence, $\eta_x(A) = \eta_x(f^{-1}(A)) \forall A \in \mathcal{G}$. Using lemma 3.5, we can conclude that $\eta_x(E) = \eta_x(f^{-1}(E))$ for every Borel set. \square

6.1. Generalized physical measures. Here we use the previous results to prove Theorem C on a necessary and sufficient condition of the existence of generalized physical measures and, as a consequence, we prove Corollary D.

Proof of first statement of Theorem C. We start by assuming that f admits a generalized μ -physical measure. Thus, there is a set $U \subset \mathbb{X}$ with $\mu(U) > 0$ s.t. $\lim_n \frac{1}{n} \sum_{j=0}^{+\infty} \varphi \circ f^j(x)$ exists for every $x \in U$ and every $\varphi \in C^0(\mathbb{X})$. So, it follows from Corollary 3.13 that η_x is a probability for every $x \in U$.

For the reciprocal, let $U = \{x \in \mathbb{X}; \eta_x(\mathbb{X}) = 1\}$ and assume that $\mu(U) > 0$. Let $\nu := \int_{x \in U} \eta_x d\mu$. As f is a continuous, it follows from Theorem 6.1 that η_x invariant $\forall x \in \mathbb{X}$. Thus, the invariance of ν follows from $\nu \circ f^{-1} = \int_{x \in U} \eta_x \circ f^{-1} d\mu = \int_{x \in U} \eta_x d\mu = \nu$. By corollary 3.13,

$$\lim_n \frac{1}{n} \sum_{j=0}^{+\infty} \varphi \circ f^j(x) = \int \varphi d\eta_x$$

for every $x \in U$ and $\varphi \in C^0(\mathbb{X})$. As a consequence,

$$\int_{x \in U} \left(\lim_n \frac{1}{n} \sum_{j=0}^{+\infty} \varphi \circ f^j(x) \right) d\mu = \int_{x \in U} \left(\int \varphi d\eta_x \right) d\mu = \int \varphi d\nu.$$

This proves the necessary and sufficient conditions for the existence of a generalized μ -physical probability measure. For the existence of a μ -physical measure we argue in Section 6.2. \square

We are now ready to present a proof of Corollary D.

Proof of Corollary D. Theorem 2.1 ensures that the generic homeomorphism $f : M \curvearrowright$ of a compact boundaryless manifold M is such that $\tilde{\varphi}(x) = \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^j(x)$ exists for every $\varphi \in C^0(M)$ and Lebesgue almost every $x \in M$. Hence, Corollary 3.13 ensures that $\tilde{\varphi}(x) = \int \varphi d\eta_x$ and η_x is a probability for Lebesgue a.e. $x \in M$. Thus we can take $\mu = \text{Leb}$ in Theorem C to conclude that f admits a Leb-generalized physical measure. \square

Example 6. In Example 2 from Bowen, we have that $2 > \eta_x(\mathbb{X}) > 1$ for all x , except the points in the homoclinic connections and the repeller fixed point at the center of region bounded by the homoclinic connections.

Hence there cannot be any positive Lebesgue measure subset where η_x is a probability. Thus this system does not admit any Lebesgue-generalized physical measure.

Example 7. Here we provide an example of a generalized μ -physical measure which is not a μ -physical measure for a continuous dynamical system. We consider the following product map $g = Id \times f : \mathbb{T}^2 \curvearrowright$, where $Id : \mathbb{S}^1 \curvearrowright$ is just the identity and $f : \mathbb{S}^1 \curvearrowright$ is the “North-South” map, that is, a map of the circle with two fixed points N, S such that N is a repeller and S an attractor and for every $p \in \mathbb{S}^1 \setminus \{N, S\}$ we have $\alpha_f(p) = \{N\}$ and $\omega_f(p) = \{S\}$.

We consider now the product probability measure $\mu = \nu \times \delta_S$ on \mathbb{T}^2 where ν is any Borel probability measure on \mathbb{S}^1 . We note that the positive orbit of $(x, y) \in \mathbb{T}^2 \setminus (\mathbb{S}^1 \times \{N\})$ converges to (x, S) and so $\eta_{(x,y)} = \tau_{(x,y)} = \delta_x \times \delta_S$; and the positive orbit of $(x, N) \in \mathbb{S}^1 \times \{N\}$ is just the point (x, N) , so that $\eta_{(x,N)} = \tau_{(x,N)} = \delta_x \times \delta_N$. Hence all points (x, y) of \mathbb{T}^2 are such that $\eta_{(x,y)}$ is a probability measure. Hence μ is a generalized μ -physical measure which is also g -invariant.

However, if we set ν to be the normalized Lebesgue measure on \mathbb{S}^1 (or any measure which is not a Dirac mass), then μ clearly is not μ -physical since the averages of Dirac masses over the positive orbit of any point do not even accumulate to μ in the weak* topology.

6.2. On non-invariant ergodic measures and physical measures. Here we present a proof of the second statement of Theorem C, on existence of μ -physical measures for continuous dynamics.

The lemma below is a characterization of the ergodic measures and the proof is analogous to the proof of the same statement for invariant ergodic measures; see, for instance [36].

Lemma 6.2. *Let μ be a measure, not necessarily a f -invariant measure. Then the following statements are all equivalent.*

- (1) μ is ergodic.
- (2) Every μ -a.e. invariant function (i.e., $\varphi \circ f = \varphi$ μ -a.e.) is constant μ -a.e..
- (3) For every measurable subset $A \subset \mathbb{X}$ the map $\mathbb{X} \ni x \mapsto \tau_x^f(A)$ is constant μ -a.e..
- (4) For every $\varphi \in L^1(\mathbb{X}, \mu)$ the map $\mathbb{X} \ni x \mapsto \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^j(x)$ is constant μ -a.e..

Moreover, if μ is ergodic, then μ does not admit a decomposition $\mu = \mu_1 + \mu_2$ where μ_1, μ_2 are finite measures such that $\mu_i \circ f^{-1} \ll \mu_i, i = 1, 2$ and $\mu_1 \perp \mu_2$ (that is, there are subsets $X_1, X_2 \subset \mathbb{X}$ such that $\mu(X_1 \cap X_2) = 0 = \mu(\mathbb{X} \setminus (X_1 \cup X_2))$ and $\mu_i(X_j) = \delta_{ij}, i, j = 1, 2$).

Now we obtain a necessary condition for a measure to be f -ergodic, even if it is not f -invariant.

Lemma 6.3. *If μ is f -ergodic (not necessarily invariant), then $\eta_x = \eta_y$ for μ -a.e. $x, y \in \mathbb{X}$.*

Proof. Let $\mathcal{A} = \{A_1, A_2, \dots\}$ be a countable basis of open sets for the topology of \mathbb{X} .

Since $\eta_x = \eta_{fx}$ for every $x \in \mathbb{X}$, if μ is ergodic then, given a measurable set $A \subset \mathbb{X}$, the function $\varphi_A : \mathbb{X} \rightarrow \mathbb{R}, x \mapsto \eta_x(A)$ is f -invariant. From Lemma 6.2 it follows that $\eta_x(A)$ is constant μ -a.e. $x \in \mathbb{X}$.

Then for each $j \in \mathbb{Z}^+$ we can find $a_j \geq 0$ such that $\mu(\mathbb{X} \setminus \varphi_{A_j}^{-1}\{a_j\}) = 0$. Thus $\mu\left(\mathbb{X} \setminus \bigcap_j \varphi_{A_j}^{-1}\{a_j\}\right) = 0$. Note that, if $x, y \in \bigcap_j \varphi_{A_j}^{-1}\{a_j\}$, then $\eta_x(A_j) = \eta_y(A_j)$ for every $j \in \mathbb{Z}^+$, and this implies that $\eta_x = \eta_y$. \square

Remark 6.4. The Bowen Example 2 satisfies the conclusion of Lemma 6.3 for $\mu =$ Lebesgue measure, but Lebesgue measure is not ergodic! Indeed, we can easily build an invariant positive Lebesgue measure subset, both for the flow X_t and for the time-1 diffeomorphism $f = X_1$, as follows.

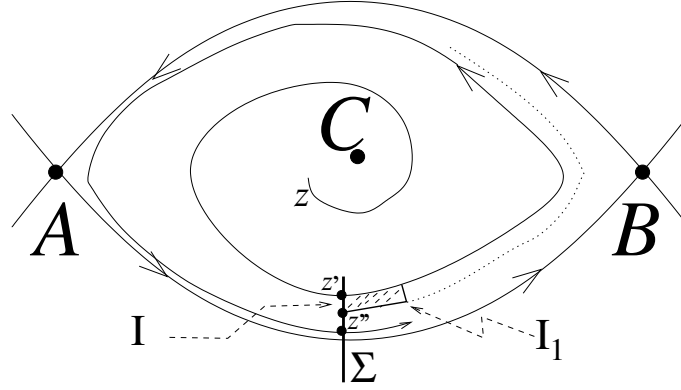


FIGURE 2. The construction in the Bowen system of a positive volume invariant subset whose complement also has positive volume.

Consider a Poincaré section Σ of one of the saddle connections and a fundamental domain of the return map given by the interval $J = [z', z'']$, as in Figure 2. Then consider an interval I strictly contained in J so that $J \setminus I$ has nonempty interior. Then the set $I_1 := X_{[0,1]}(I)$ is such that $W = \bigcup_{n \in \mathbb{Z}} f^n I_1$ is both f -invariant and X_t -invariant for every $t \in \mathbb{R}$. Moreover, by construction, both W and its complement have positive Lebesgue measure (area).

We now provide necessary and sufficient conditions for a measure μ to admit a μ -physical measure, even if μ is not f -invariant.

Theorem 6.5. *If $f : \mathbb{X} \rightarrow \mathbb{X}$ is continuous then the statements below are all equivalent.*

- (1) f admits a μ -physical measure.

- (2) *There exists $U \subset \mathbb{X}$ such that $\mu(U) > 0$ and*
 (a) *$\mu|_U$ is an ergodic measure (not necessarily invariant), and*
 (b) *$\eta_x(\mathbb{X}) = 1$ for each $x \in U$.*
 (3) *There exists $U \subset \mathbb{X}$ with $\mu(U) > 0$ such that $\eta_x(\mathbb{X}) = 1$ and $\eta_x = \eta_y$ for every $x, y \in U$.*

Proof. (1) \Rightarrow (2). Suppose that f admits a μ -physical measure ν . Thus, there is a set $U \subset \mathbb{X}$ with $\mu(U) > 0$ so that $\lim_n \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^j(x) = \int \varphi d\nu$ exists for every $x \in U$ and every $\varphi \in C^0(\mathbb{X})$. So, it follows from Corollary 3.13 that, for every $x \in U$, η_x is a probability measure. Moreover, $\int \varphi d\eta_x = \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^j(x) = \int \varphi d\nu$ for each $\varphi \in C^0(\mathbb{X})$ and for every $x \in U$. Hence $\eta_x = \nu$ for each $x \in U$. As a consequence, the ergodicity of μ follows from Lemma 6.3.

(2) \Rightarrow (3) is immediate from lemma 6.3.

(3) \Rightarrow (1). Now, assume that $U = \{x \in \mathbb{X}; \eta_x(\mathbb{X}) = 1\}$ has positive μ measure and $\eta_x = \eta_y$ for every $x, y \in U$. Let us define the probability measure $\nu := \frac{1}{\mu(U)} \int_{x \in U} \eta_x d\mu$. By Theorem 6.1, η_x is an f -invariant measure for each $x \in \mathbb{X}$. Hence $\nu \circ f^{-1} = \frac{1}{\mu(U)} \int_{x \in U} \eta_x \circ f^{-1} d\mu = \frac{1}{\mu(U)} \int_{x \in U} \eta_x d\mu = \nu$ is an f -invariant probability measure. As the function $U \ni x \mapsto \eta_x$ is constant, it follows that $\eta_x = \nu$ for every $x \in U$. Thus, by Corollary 3.13,

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^j(x) = \int \varphi d\eta_x = \int \varphi d\nu$$

for every $x \in U$ and $\varphi \in C^0(\mathbb{X})$. The proof is complete. \square

Proof of the second statement of Theorem C. The necessary and sufficient condition for the existence of a μ -physical probability measure in Theorem C is item (2) of Theorem 6.5. This, together with Section 6.1, concludes the proof of Theorem C. \square

7. ON STRONG HISTORIC BEHAVIOR

Here we prove Theorem E obtaining a partial characterization of the behavior of Example 2. We start by observing that if η_x is an atomic measure, then the atoms must be preperiodic points of the transformation.

Lemma 7.1. *If η_x is purely atomic for some $x \in X$, with $f : X \rightarrow X$ a continuous map with finitely many pre-images and X a perfect metric space, then every atom P is a preperiodic point for f . In particular, if f is a bijection, then P is a periodic point for f .*

Proof. From Theorem 6.1 we know that η_x is a f -invariant measure. Then $\eta_x(f(P)) = \eta_x(f^{-1}(f(P))) \geq \eta_x(P) > 0$ and so $f(P)$ is an atom of η_x . Since we assume that η_x has finitely many atoms, then f maps this finite set into itself, and each atom is preperiodic for f . \square

Let us assume that the system $f : M \rightarrow M$ is a diffeomorphism on a compact boundaryless manifold and that there are at least a pair of hyperbolic periodic points P, Q of f satisfying, for some $\varepsilon > 0$

- (H) η_x is atomic with two atoms P, Q for every x either in $B_\varepsilon(P) \setminus (W_\varepsilon^s(P) \cup W_\varepsilon^u(P))$ or $B_\varepsilon(Q) \setminus (W_\varepsilon^s(Q) \cup W_\varepsilon^u(Q))$.

We will show that under these conditions we have the same configuration as in Example 2.

Theorem 7.2. *Under assumption (H) we have that $W^s(P) = W^u(Q)$ and $W^u(P) = W^s(Q)$, that is, P and Q have heteroclinic connections.*

Proof. Let us fix $\varepsilon > 0$ with the properties given in assumption (H) and x such that $\eta_x = a\delta_P + b\delta_Q$ with $a, b > 0$.

The assumption that η_x is formed by precisely two atoms ensures that P, Q are fixed points, for otherwise all the points in the orbit of P, Q would also be atoms of η_x .

The definition of η_x ensures that $\tau_x(B_\delta(P)) > 0$ for all $\delta > 0$. Hence we can find a sequence k_i of iterates such that $x_{k_i} := f^{k_i}(x) \rightarrow P$ when $i \rightarrow +\infty$. Since P is a hyperbolic fixed point, there exists $n_k \nearrow \infty$ and a point $y \in W_{\varepsilon/2}^s(P)$ such that $x_{n_k} \rightarrow y$ when $k \rightarrow \infty$.

Writing $\sigma > 1$ for the least expanding eigenvalue of $Df(P)$ and $\lambda = \sup \|Df\|$ we obtain for some $m_k \nearrow \infty$

$$\text{dist}(x_{n_k}(x), y) \leq \sigma^{-m_k} \quad \text{and} \quad \text{dist}(x_{n_k-i}(x), y_{-i}) \leq \sigma^{-m_k} \lambda^i, \quad 0 \leq i \leq n_k. \quad (9)$$

We claim that there exists $\ell \geq 1$ such that $y_{-\ell} \in B_\varepsilon(Q)$.

Indeed, let us assume that $y_{-\ell} \notin B_\varepsilon(Q)$ for each $\ell \geq 1$. Hence, for every big enough k , each visit x_{n_k} to $B_\varepsilon(P)$ corresponds to a visit to $K = B_\varepsilon(W_\varepsilon^s(P)) \setminus (B_\varepsilon(P) \cup B_\varepsilon(Q))$ of $x_{n_k-\ell}$ for some $\ell \geq 1$; see Figure 3. Moreover from (9) we can ensure that the set of ℓ for which $x_{n_k-\ell}$ visits K has size proportional to m_k for big k , that is

$$\frac{1}{m_k} \max\{\ell \geq 1 : \sigma^{-m_k} \lambda^\ell < \varepsilon\} \xrightarrow{k \rightarrow \infty} -\frac{\log \sigma}{\log \lambda} = \xi.$$

Thus we obtain that $\tau_x(M \setminus (B_\varepsilon(P) \cup B_\varepsilon(Q))) \geq \tau_x(K) \geq \xi \tau_x(B_\varepsilon(P)) > 0$ which contradicts the assumption that the atoms of η_x are P, Q only. This proves the claim.

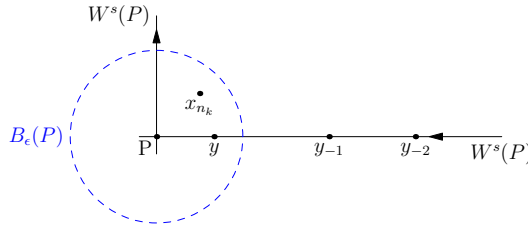


FIGURE 3. The position of y and y_ℓ around P .

We note that $y_{-\ell} \in B_\varepsilon(Q)$ does not satisfy (H) unless $y_{-\ell} \in W^u(Q) \cup W^s(Q)$. But because $y \in W^s(P)$ we deduce that $y_{-\ell} \in W^u(Q)$.

Moreover, see Figure 4, we cannot have a transversal intersection between $W^u(Q)$ and $W^s(P)$ at $y_{-\ell} \in B_\varepsilon(Q)$, for otherwise we would obtain a point $z \in W^s(P) \cap B_\varepsilon(Q)$ which contradicts (H). By the same reason, a tangency between $W^u(Q)$ and $W^s(P)$ at $y_{-\ell} \in B_\varepsilon(Q)$ is not allowed. Hence the connected component of $W^s(P)$ in $B_\varepsilon(Q)$ containing $y_{-\ell}$

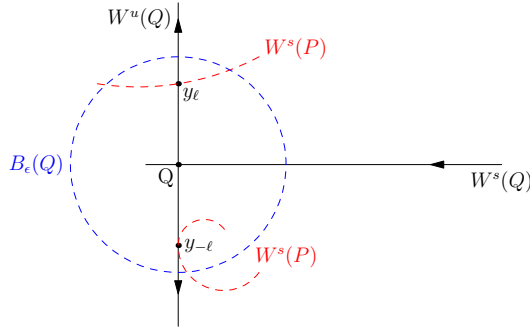


FIGURE 4. The position of $y_{-\ell}$ and $W^s(P)$ near Q ; in the upper side the transversal situation, in the lower side the tangent situation.

must be contained in $W^u(Q)$. Thus, $W^u(Q) \subset W^s(P)$ since both invariant manifolds are immersed submanifolds of M and they are uniquely defined in a neighborhood of P, Q .

An analogous reasoning provides $W^u(P) \subset W^s(Q)$. Since the dimensions of the stable and unstable manifolds of each P, Q are complementary, we conclude that the dimensions of $W^s(P)$ and $W^u(Q)$ are the same, and so $W^s(P) = W^u(Q)$. Similarly we arrive at $W^u(P) = W^s(Q)$. \square

Remark 7.3. The argument in the proof of Theorem 7.2 can be easily adapted to a setting where the number of atoms of η_x is a finite set of hyperbolic periodic points for all x in a neighborhood of each periodic point such that x leaves that neighborhood in the future and past. That is, for all x in a neighborhood of the periodic points with the exception of its invariant manifolds. We then obtain heteroclinic connections between a finite family of hyperbolic periodic points as in [22].

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